An Introduction to

DEDUCTIVE LOGIC

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HUGUES LEBLANC
ASSOCIATE PROFESSOR OF PHILOSOPHY
BRYN MAWR COLLEGE

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To the memory of my father

Preface

This book is offered as a manual for advanced undergraduate and elementary graduate work in logic. It includes some of the fundamentals of modern logic; it excludes, however, as much as, if not more than, it includes. A short bibliography will be found at the back of the book, several items of which are a must for philosophy, mathematics, and science students.

The material (exclusive of starred paragraphs) can easily be covered in a one-semester course, each chapter requiring approximately a month's work. The fifth chapter is somewhat more technical and may be left out in undergraduate courses. A series of problems is offered at the end of the book for classroom and home work.

The method of proof used here is believed to be both rigorous and easy to master. More streamlined methods, though available, have been purposely ignored. Instructors are asked, however, to devise shortcuts of their own if and when the student is ripe for them.

Symbols, both logical and metalogical, have been used aplenty. Their presence may somewhat slow down the reading of the text, but is justified, we feel, in a primer of symbolic logic. The more philosophical implications of logic, though suggested throughout the text, have not been studied in full. It is hoped again that the student will peruse the list of readings to complete his own picture of logic as a province of philosophy.

The material of this book has been used in various courses at Bryn Mawr College and Columbia University. Though the product of some years of teaching, it is plainly open to improvement. The author only hopes that it may equip some philosophy, mathematics, and science students with the indispensable tools of modern logic.

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I owe to the writings of Professors R. Carnap, A. Church, S. C. Kleene, W. V. Quine, and J. B. Rosser much of my knowledge of logic and hence much of the material of this book.

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INTRODUCTION

1. SEMIOTIC AND LOGIC

Signs are gestures, sounds, or inscriptions used by man to communicate with himself or with his fellow men. A smile, for instance, is a gesture often used to show friendliness; a moan is a sound often used to express pain; and 'explosives' is an inscription often used to warn of impending danger. Sounds may vary in length from phonemes to spoken words and spoken statements; similarly, inscriptions may vary in length from letters to written words and written statements. We shall be mostly concerned here with words and statements.

Words are very plastic means of communication. They may serve to convey a wish; the verb 'get' in the statement:

If only Annie could get her gun,

is used, for instance, to convey a wish. They may also serve to convey an order; the verb 'get' in the statement:

Annie, get your gun,

is used, for instance, to convey an order. They may finally serve to convey information; the verb 'get' in the statement:

Annie gets her gun,

is used, for instance, to convey information.

Words, when they convey a wish, are said to function *appraisively*; when they convey an order, to function *prescriptively*; and when they convey information, to function *descriptively*. The reader will find that on the average words function appraisively when they occur in subjunctive statements; prescriptively when they occur in imperative statements; and descriptively when they occur in indicative statements.[1]

The general study of signs has been labelled by John Locke *semiotic* and the part of semiotic dealing with descriptive signs labelled by the

[1]A distinction is sometimes drawn among signs between symbols and words, symbols belonging to technical languages, words to non-technical ones; it is not officially adopted here.

Greeks *logic.* We shall use Locke's 'semiotic' as our official tag for the study of signs; in section 5, however, we shall rechristen the study of descriptive signs.

2. OBJECT LANGUAGES VERSUS METALANGUAGES

We may use an object or mention it; similarly, we may use a sign or mention it. We often use a sign to talk about what it designates; in the statement:

Trenton is in New Jersey,

for instance, we use the word 'Trenton' to talk about the city Trenton which it designates. We always mention a sign to talk about the sign itself; in the statement:

'Trenton' is made up of seven letters,

for instance, we mention the word 'Trenton' to talk about the word 'Trenton' itself.

It is hard to mistake an object which is mentioned for one which is used; it is easy, however, to mistake a sign which is mentioned for one which is used. To prevent the confusion, we shall enclose all mentioned signs, unless displayed on a separate line, within quotation marks. We shall thus write:

'Trenton' is made up of seven letters,

rather than:

Trenton is made up of seven letters;

we shall, however, take the liberty of writing:

'Trenton' is made up of seven letters,

rather than:

' 'Trenton' is made up of seven letters'.

Any number of quotation marks may enclose a given sign. As we wrote:

'Trenton' is made up of seven letters,

so we shall write:

''Trenton'' is made up of seven letters and one pair of quotation marks,
'''Trenton''' is made up of seven letters and two pairs of quotation marks,

and so on. Semiotic quotes, as we may informally call them, are a name-forming operator; they should not be confused with rhetorical quotes, which simply help stressing words.

As we must mention a word if we are to talk about it, so we must

mention a statement if we are to talk about it. Consider, for instance, the statement:

Trenton is south of Boston (1).

If we wish to say that (1) is true, we must mention (1) within our intended statement:

'Trenton is south of Boston' is true (2);

and if we wish to say that (1) is made up of five words, we must mention (1) within our intended statement:

'Trenton is south of Boston' is made up of five words (3).

(2) and (3), being statements about a statement, may be called *metastatements*; (1), on the other hand, being the object of statements (2) and (3), may be called an *object statement.*

These preliminary distinctions lead to the distinction between an object language and a metalanguage. Let a language L be given. L will include words like 'Boston', 'Trenton', 'New Jersey', 'is', 'south', 'in', 'of', and so on, and statements made up of these words like:

Trenton is south of Boston,
Trenton is in New Jersey,

and so on. The words and statements of L may either be used or be mentioned. If mentioned, they will become the object of a discourse, a discourse carried out like all discourses in a language, say language L'. The language L about which the discourse is carried out will be called an *object language* and the language L' within which the discourse is carried out will be called a *metalanguage*. The two metastatements (2) and (3) thus belong to the metalanguage L' of the above sample language L.

Since a given metalanguage may be mentioned in a discourse of its own, it may become the object language of a second metalanguage, which may become in turn the object language of a third metalanguage, and so on, *ad infinitum*. The metalanguage within which semiotic is carried out, for example, may become the object language of a second metalanguage, which may become in turn the object language of a third metalanguage, and on, *ad infinitum*. Because of this piling up of metalanguages over metalanguages, semiotic (and, hence, logic) is ultimately carried out, not in one metalanguage, but in an infinite set of metalanguages.

3. THE THREE DIMENSIONS OF SEMIOTIC

Semiotic has three dimensions: *syntax*, *semantics*, and *pragmatics*. We shall first define semantics and pragmatics, and then turn to syntax.

Semantics studies the various relations holding between signs and whatever signs may serve to discourse about; it studies, for instance, the relation *designating*, the relation *being true of*, and so on. It also studies various properties of signs like the property *being true*, the property *being valid*, and so on, which are definable in terms of the above relations. As sample semantical statements we may quote:

The expression 'The bard of Avon' designates Shakespeare,
The predicate 'is black' is true of coal, tar, and so on,
The statement 'Coal is black' is true if and only if coal is black,

and so on.

Pragmatics studies the various relations holding between signs and sign users; it studies, for instance, the relation *being meaningful to*, the relation *being known to be true by*, and so on. It also studies various properties of signs like the property *being in use*, the property *being believable*, and so on, which are definable in terms of the above relations. As sample pragmatical statements we may quote:

The word 'duty' is not meaningful to a three-year old child,
The word 'semiotic' has been in use since John Locke,
The statement 'Mars will invade us in 1970' is hardly believable,

and so on.

Semantics and pragmatics treat signs as interpreted items of discourse. Syntax, on the other hand, ignoring the interpretation placed upon discourse by semantics and pragmatics, treats signs as shapes, spoken shapes or written ones; it studies the various relations holding from a sign to itself or to another sign, like the relation *being the first letter of*, and the properties definable in terms of these relations, like the property *being a five-letter word*. As sample syntactical statements we may quote:

'L' is the first letter of the word 'Logic',
'Logic' is a five-letter word,

and so on.

Syntax is clearly basic to semantics and pragmatics; though the first, it is not, however, as the Positivists once claimed, the only dimension of semiotic. If signs are to serve as instruments of communication, they must be signs of something for someone; studying them as signs of something falls by definition to semantics and studying them as signs for someone falls by definition to pragmatics. Both semantics and pragmatics must therefore be acknowledged as integral dimensions of semiotic.[1a]

[1a]Semantics and pragmatics are often merged together under the label "semantics'; when this is done, syntax may be characterized as the study of uninterpreted signs, semantics as the study of interpreted ones.

4. THE ANATOMY OF DESCRIPTIVE LANGUAGES

Let us now turn to the descriptive languages of which logic is said to be the study. These languages form the network of signs in which is couched our empirical knowledge of things. In order to classify them we may first survey the various provinces of science; they are, roughly:

1. physics, and all the disciplines related to physics like chemistry, astronomy, and geology;
2. biology, and all the disciplines related to biology like anatomy, physiology, and zoology;
3. psychology and the social sciences, grouped here under the convenient heading 'behavioristics'.

Since each science is expressed in a language of its own, we get three classes of descriptive languages: physical, biological, and behaviorial languages.

We spoke above of provinces of science; the metaphor proves to be inadequate, for if physics is independent of biology, and biology independent of behavioristics, biology is nevertheless founded on physics, and behavioristics founded on biology. We thus gather that each science is a layer, not a province, of human knowledge.

The same relations hold between descriptive languages; the language of physics is part of the language of biology and the language of biology part of the language of behavioristics. This nesting of descriptive languages may best be exemplified in connection with words. Behavioristics, to describe individual and social behavior, uses words which are proper to behavioristics; but it also carries over the vocabulary of biology. Biology, in turn, to describe life, uses words which are proper to biology; but it also carries over the vocabulary of physics.

But what about physics? Is its vocabulary entirely physical? No, it includes words like 'mass', 'energy', 'motion', which belong to physics proper, and words like 'or', 'all', 'is', which physics seems to borrow from a previous language. The latter words pervade the whole of physics, biology, and behavioristics; they are basic to all descriptive languages, and yet do not belong to any descriptive language in particular. We shall call them *logical words* (in a new sense of the word 'logical') and call the words proper to each descriptive language *factual words*.

The sorting out of logical words is somewhat arbitrary; it usually includes, however, conjunctions like 'or', 'and', and so on, indefinite adjectives like 'all', 'some', and so on, and predicates like 'is identical with', 'is a member of', and so on. These words may be viewed as constituting a language-form which, once filled out with factual words, yields

the languages of physics, biology, and behavioristics. The language-form in question, due in part to Aristotle, in part to the Stoics, in part to the Scholastics, and in part to mathematicians of the nineteenth century, is often called *logic*. To prevent confusion we shall refer to the study of descriptive languages as *logic*$_1$ and to the framework of all descriptive languages as *logic*$_2$.

If logic$_2$ is the framework of all descriptive languages, it is also the framework of all metalanguages. Semiotic uses indeed, to discourse about languages, two sorts of words:

(*a*) semiotic words like 'designates', 'meaningful', and so on;

and

(*b*) logical words like 'or', 'all', 'is', and so on.

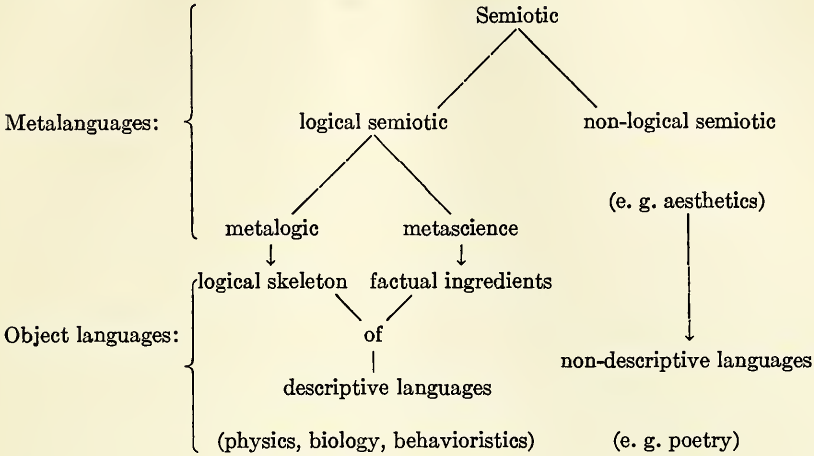
Logic$_2$ may thus be identified as the framework of most languages, whether object languages or metalanguages.

In our present anatomy of science we have not mentioned mathematics. Mathematics was for centuries regarded as an independent discipline, somewhat parallel to logic$_2$ and hence also basic to natural science. It was assigned a mythical subject matter: quantity (either discrete or continuous); its laws were pronounced *synthetic a priori*, the synthetic *a priori* standing midway between the logically true (or *a priori*) and the factually true (or synthetic); and its methods were applied throughout science on the ground that numbers and things are somehow germane. Deeper variants of this metamathematics were set forth, and hence more cryptic ones. The issue was clarified only with the advent of modern logic, which showed that mathematics reduces to logic; in other words, that mathematical signs are definable in terms of logical ones and mathematical truths deducible from logical ones. This result, due essentially to the German logician Gottlob Frege (1848–1925), was restated by A. N. Whitehead and B. Russell in their epoch-making *Principia Mathematica*.

5. THE CHRISTENING OF LOGIC

We have disentangled in the foregoing sections two things which go by the name of logic and labelled them, for convenience's sake, *logic*$_1$ and *logic*$_2$. We must now devise for them more appropriate tags. Logic$_2$, which we identified as the framework of all descriptive languages, will be called *logic*; logic$_1$, which we identified as the study of all descriptive languages, will be called *logical semiotic*. One part of logical semiotic deals with the logical ingredients of descriptive languages; we may call it *metalogic*. Another part deals with their factual ingredients; we may call it *metascience*.

The following diagram will help to sum up the distinctions drawn so far:



Logic falls into two main parts: *elementary logic* and *general logic.*[2] We shall restrict ourselves in this introduction to the former. Chapter one will cover a fragment of elementary logic known as *sentential logic*; chapter two, another fragment known as *quantificational logic.* Chapter three will formalize both logics into so-called *calculi*; chapter four, cover a last fragment of elementary logic known as *the logic of identity, classes, and relations*; and chapter five, present sample excerpts from the syntax of elementary logic.[3]

Of the various problems which pertain to a study of elementary logic one is treated here in full: *the problem of deduction.* Its twin, *the problem of induction,* has been reserved for another book, planned under the title: *An Introduction to Inductive Logic.*

[2]General logic is built around the predicate 'is a member of'; it soon merges, as noted above, with mathematics.

[3]Studies on the semantics and pragmatics of elementary logic are listed in the bibliography.

CHAPTER

O N E

Sentential Logic

6. SENTENTIAL LETTERS, CONNECTIVES, AND PARENTHESES

Our study of language and of the language-form called *logic* will focus on statements. We shall first learn how to *combine* statements with the help of conjunctions (chapter one); we shall next learn how to *form* statements out of predicates, substantives, and indefinite adjectives (chapter two). The fragment of logic studied in this chapter is often termed *sentential logic* or, more specifically, *two-valued sentential logic.* The choice of the first label is self-explanatory; the choice of the second will soon be defended.

Indicative statements fall into two groups: *atomic* statements and *molecular* ones. Atomic statements are statements like:

John is elected (1)

and

Mary will rejoice (2),

which do not include any conjunction. Molecular statements, on the other hand, are compounds of atomic statements and conjunctions like:

If John is elected, then Mary will rejoice (3)

and

Mary will rejoice if and only if John is elected (4);

(3), as the reader may check, is made up of the two atomic statements (1) and (2) and the conjunction 'if-then'; (4), made up of the two atomic statements (1) and (2) and the conjunction 'if and only if'.

We shall use the four letters 'p', 'q', 'r', and 's' to serve as place-holders for atomic and molecular statements generally; the letters in question will be called *sentential letters* and the statements for which they stand will be called their *instances.* We shall also use compounds of sentential letters and conjunctions like:

If p, then q (5)

and

$$p \text{ if and only if } q \qquad (6),$$

to serve as place-holders for molecular statements. (5), for example, will admit as its instances all the molecular statements which may be obtained by substituting statements for the two letters 'p' and 'q' of (5); one of these instances will be (3) above, which may be obtained by substituting 'John is elected' for 'p' and 'Mary will rejoice for 'q' in 'If p, then q'. (6), on the other hand, will admit as its instances all the molecular statements which may be obtained by substituting statements for the two letters 'p' and 'q' of (6); one of these instances will be (4) above, which may be obtained by substituting 'Mary will rejoice' for 'p' and 'John is elected' for 'q' in 'p if and only if q'.

Sentential letters and compounds of sentential letters and conjunctions will be grouped under the generic heading 'sentential schemata'; sentential schemata and statements, under the generic heading 'sentential formulae'.

Conjunctions or *connectives*, as we technically call them, may govern either one formula or two formulae at a time. When they govern one formula at a time, they are termed *singulary*; when they govern two formulae at a time, they are termed *binary*. One singulary and five binary connectives deserve attention.

The singulary connective is 'not'. This particle, in everyday language, occurs mostly within statements; in logic, however, it is prefixed to statements. 'John is not sick' will thus become 'Not(John is sick)' or, more idiomatically, 'It is not the case that John is sick'. We shall translate 'not' into '$\sim$' and call the result of writing '$\sim$' in front of a formula a *denial* or *negation*. We shall take the denial of a statement to be true when the statement in question is false, and to be false when the statement in question is true.

Our first binary connective is 'and', which we translate by a dot. The result of inserting '.' between two formulae is called a *conjunction*; its two components are called *factors*, because of the analogy holding between '.' and the times sign of algebra. To quote at random, 'Peter is jealous . Mary is frivolous' is a conjunction; its two factors are the atomic statements 'Peter is jealous' and 'Mary is frivolous'. We shall take a conjunction to be true when its factors are both true, and to be false when at least one of its factors is false.

Our second and third binary connective is 'or'. We say: "second and third," because 'or' is used in everyday language in two different contexts:

1. the context 'p or q', sometimes reinforced to read 'p and/or q';

2. the context 'either p or q', sometimes reinforced to read: 'p or q but not both'.

We shall call the first context an *inclusive alternation* and translate it into '$p \vee q$'; we shall call the second context an *exclusive alternation* and translate it into '$p \not\equiv q$'. As an instance of '$p \vee q$' we may quote:

John will visit France $\vee$ John will visit Italy,

that is,

John will visit France and/or Italy;

as an instance of '$p \not\equiv q$' we may quote:

John will visit France $\not\equiv$ John will visit Italy,

that is,

John will visit France or Italy but not both.

The two components of an inclusive alternation will be called *summands*, because of the analogy holding between '$\vee$' and the plus sign of algebra. We shall take an inclusive alternation to be true when at least one of its summands is true, and to be false when its summands are both false. We shall take an exclusive alternation to be true when one of its components is true and the other one false, and to be false when its components are either both true or both false.

Our fourth binary connective, 'if-then', translates into '$\supset$'. The result of inserting '$\supset$' between two formulae is called a *conditional*; the formula preceding '$\supset$' is called the *antecedent* of the conditional, the formula following '$\supset$' its *consequent*. 'Harry is promoted $\supset$ James is demoted' is a sample conditional, with 'Harry is promoted' as antecedent and 'James is demoted' as consequent. We shall take a conditional to be true when its antecedent is false or its consequent true, and to be false when its antecedent is true and its consequent false.

This last convention may puzzle the reader. Conditionals with true antecedents frequently occur in everyday discourse and are declared true or false depending upon whether their consequent is true or false.[1] But conditionals with false antecedents rarely occur in everyday discourse; we accordingly have no standard by which to declare them true or false. The convention we adopt here, besides being handy, is motivated by the following consideration. A conditional may be said to be true if it is not the case that its antecedent is true and its consequent false, hence if the conjunction made up of its antecedent and the denial of its consequent is false; but the conjunction in question is false whenever its first factor

[1]Conditionals with true antecedents are often phrased 'Since ——, then . . .' rather than 'If——, then . . .'.

is false; hence a conditional whose antecedent is false may be said to be true (whether, let us add, its consequent be true or false).[2]

Our last connective, 'if and only if', translates into '$\equiv$'. The result of inserting '$\equiv$' between two formulae is called a *biconditional*. 'Harry is promoted $\equiv$ James is demoted', for instance, is a biconditional; it is equivalent, like all biconditionals, to a conjunction of two conditionals, namely: '(Harry is promoted $\supset$ James is demoted) . (James is demoted $\supset$ Harry is promoted)'. We shall take a biconditional to be true when its components are both true or both false, and to be false when one of its components is true and the other one is false.

We have identified above six sentential schemata: '$\sim p$', '$p \,.\, q$', '$p \vee q$', '$p \not\equiv q$', '$p \supset q$', and '$p \equiv q$', respectively read: 'not p', 'p and q', 'p or q', 'either p or q', 'if p, then q', and 'p if and only if q'. Everyday language has many variants for these readings; some are appended:

for 'p and q': 'p, but q', 'p, although q', 'p, q';
for 'if p, then q': 'if p, q', 'q, if p', 'q provided that p', 'q granted that p'.

A last variant calls for discussion: 'not p unless q'. It may be interpreted in two different ways: either as '$p \equiv q$' or as '$p \supset q$'. The statement: '10 is not divisible by 2 unless 10 is divisible by $\sqrt{4}$', clearly translates into: '10 is divisible by 2 $\equiv$ 10 is divisible by $\sqrt{4}$'. The statement: 'A judge is not appointed to the Supreme Court unless he is well grounded in law', must, on the other hand, be phrased as: 'A judge is appointed to the Supreme Court $\supset$ he is well grounded in law'. The biconditional: 'A judge is appointed to the Supreme Court $\equiv$ he is well grounded in law', being equivalent to the conjunction: '(A judge is appointed to the Supreme Court $\supset$ he is well grounded in law) . (A judge is well grounded in law $\supset$ he is appointed to the Supreme Court)', would indeed be stronger than our original statement.

The connectives that we have been studying so far are statement connectives; some noun connectives, however, are nothing but disguised statement connectives, and the atomic statements in which they occur can easily be paraphrased as standard molecular ones. Such is the case with

Mary and John came,

which expands into:

[2]The only conditionals we are studying here are indicative ones. Subjunctive conditionals with false antecedents may, as opposed to indicative ones, be false; 'If Truman had run for the presidency in 1952, he would have been elected', for instance, is presumably false. Some subjunctive conditionals do, by the way, serve a descriptive rather than an appraisive purpose; attempts to translate them into indicative conditionals have so far failed.

Mary came and John came,

and

Mary or John came,

which expands into:

Mary came or John came.

Genuine noun connectives, such as the 'and' of 'Canada and Great Britain have agreed on a mutual defense pact', will be studied later.

We may now turn to the last two signs of sentential logic: '(' and ')'. When we translated 'John is not sick' into standard notation, we enclosed 'John is sick' within parentheses to show that '$\sim$' governs these three words as a whole. Such a grouping device is useless when '$\sim$' governs a single letter, but it is essential whenever '$\sim$' governs a compound expression. 'It is not the case that p and q', for instance, must be translated into '$\sim(p \,.\, q)$'. The scope of '$\sim$' here is the sentential compound '$p \,.\, q$' and this fact must be emphasized by parentheses; '$\sim p \,.\, q$', on the other hand, would read: 'It is not the case that p and (it is the case that) q'.

Parentheses will be used for another purpose. A binary sentential compound is made up of one connective and two components; each one of these components may in turn be a compound, built around a connective of its own, and so on. For example,

If Harry is promoted, then John is demoted, and if John is demoted, then Harry is promoted (7),

is a conjunction whose factors are two conditionals. When a given compound enters another compound, the connective around which it is built is called a *minor connective*; such are the if-then signs of (7). When, on the contrary, a given compound enters no other compound, the connective around which it is built is called a *major connective*; such is the 'and' of (7). We may agree to enclose within parentheses all binary compounds showing a minor connective. (7) will thus translate into:

(Harry is promoted $\supset$ John is demoted) . (John is demoted $\supset$ Harry is promoted).

Parentheses are clearly essential at this point. (7), we know, is an instance of

$$(p \supset q) \,.\, (q \supset p) \qquad (8);$$

yet the sentential schemata:

$$p \supset ((q \,.\, p) \supset p),$$
$$p \supset (q \,.\, (q \supset p)),$$
$$((p \supset q) \,.\, q) \supset p,$$

though distinct from each other and from (8), are all groupings of the same string of signs: 'p', '$\supset$', 'q', '.', 'q', '$\supset$', and 'p'. Parentheses play in this connection the role assumed by punctuation in everyday writing.

7. SELECTED SENTENTIAL SCHEMATA

Some sentential schemata, like:

$$p \supset p,$$

yield only truths, whatever statements be substituted for their letters; others, like:

$$p \supset q,$$

yield both truths and falsehoods, depending on the statements substituted for their letters; others, finally, like:

$$\sim(p \supset p),$$

yield only falsehoods, whatever statements be substituted for their letters. Schemata of the first type are technically said to be *sententially valid* and their instances are technically said to be *sententially true*. Sentential truths are logical truths of a sort; we shall encounter in chapter two logical truths of another sort, namely: *quantificational truths*.

Sentential truths may be negations like:

$$\sim(\text{John is sick} \;.\; \sim(\text{John is sick}));$$

conjunctions like:

$$(\text{John is sick} \supset \text{John is sick}) \;.\; (\text{James is mad} \vee \sim(\text{James is mad}));$$

alternations like:

$$\text{James is mad} \vee \sim(\text{James is mad})$$

and

$$\text{James is mad} \not\equiv \sim(\text{James is mad});$$

conditionals like:

$$(\text{Germany is south of Sweden} \;.\; \text{Italy is south of Germany}) \supset \text{Italy is south of Germany};$$

or biconditionals like:

$$\text{Italy is south of Germany} \equiv \sim\sim(\text{Italy is south of Germany}).$$

The last two possibilities deserve special notice.

Let us first compare the two statements:

$$\text{Germany is south of Sweden} \;.\; \text{Italy is south of Germany} \qquad (1)$$

and

$$\text{Italy is warmer than Sweden} \qquad (2),$$

with the two statements:

Germany is south of Sweden . Italy is south of Germany (3)

and

Italy is south of Germany (4).

Both (1) and (2) are true; yet (1) could be true without (2) being also true. If, for instance, Italy did not happen to be warmer than Sweden, then the statement:

Germany is south of Sweden . Italy is south of Germany,

could be true without the statement:

Italy is warmer than Sweden,

being also true. Take, however, (3) and (4); (3) cannot be true without (4) being also true.

Whenever a given statement cannot be true without another statement being also true, we shall say that the former *logically implies* the latter. But the antecedent of a sententially true conditional cannot be true without its consequent being also true; we may therefore say of a sententially true conditional that its antecedent logically implies its consequent. We shall likewise see in chapter two that the antecedent of a quantificationally true conditional cannot be true without its consequent being also true, and hence that the antecedent of a quantificationally true conditional logically implies its consequent. To distinguish, when convenient, between the two cases, we shall say that the antecedent of a sententially true conditional *sententially implies* its consequent, whereas the antecedent of a quantificationally true conditional *quantificationally implies* its consequent.

Let us next compare the two statements:

Italy is south of Germany (5)

and

Italy is warmer than Germany (6),

with the two statements:

Italy is south of Germany (7)

and

$\sim\sim$(Italy is south of Germany) (8).

Both (5) and (6) are true; yet (5) could be true without (6) being true and (6) could be true without (5) being true. Take, however, (7) and (8); (7) cannot be true without (8) being true and (8) cannot be true without (7) being true.

Whenever a given statement cannot be true without another statement being also true and *vice-versa,* we shall say that the two statements are *logically equivalent.* But the left-hand component of a sententially true biconditional cannot be true without its right-hand component being also true and *vice-versa*; we may therefore say of the two components of a sententially true biconditional that they are logically equivalent. We shall likewise see in chapter two that the left-hand component of a quantificationally true biconditional cannot be true without its right-hand component being also true and *vice-versa,* and hence that the two components of a quantificationally true biconditional are logically equivalent. To distinguish, when convenient, between the two cases, we shall say that the two components of a sententially true biconditional are *sententially equivalent,* whereas the two components of a quantificationally true biconditional are *quantificationally equivalent.*

We next give a list of 48 sententially valid schemata; most of them, as expected, are conditional or biconditional schemata. The reader should familiarize himself with the list by constructing instances of each and every schema.

T1: $p \vee \sim p$;
T2: $\sim(p \,.\, \sim p)$.

T1, called *the law of the Excluded Middle,* and T2, called *the law of Non-Contradiction,* were both known to Aristotle and played a crucial part in Aristotelian logic.

T3a: $p \supset p$;
T3b: $p \equiv p$.

T3a and T3b are known as *laws of Reflexivity*; they show that a statement logically implies itself and is logically equivalent to itself.

T4a: $(p \,.\, p) \equiv p$;
T4b: $(p \vee p) \equiv p$.

T4a and T4b, known as *laws of Idempotence,* show that the result of conjoining or of alternating a statement with itself is logically equivalent to that statement.

T5: $p \equiv \sim\sim p$.

T5, known as *the law of Double Negation,* shows that a statement is logically equivalent to the negation of its own negation.

T6a: $(p \,.\, q) \supset p$;
T6b: $(p \,.\, q) \supset q$;
T7a: $p \supset (p \vee q)$;
T7b: $q \supset (p \vee q)$.

T6a–T7b are called *laws of Simplification*; they show that conjunctions logically imply their factors and that alternations are logically implied by their summands.

The next three laws, called *laws of Commutativity*, show that the ordering of components in a conjunction, an alternation, and a biconditional is immaterial:

T8a: $(p \,.\, q) \equiv (q \,.\, p)$;
T8b: $(p \vee q) \equiv (q \vee p)$;
T8c: $(p \equiv q) \equiv (q \equiv p)$.

The next three laws, called *laws of Associativity*, show that the grouping of components in a conjunction, an alternation, and a biconditional is immaterial:

T9a: $((p \,.\, q) \,.\, r) \equiv (p \,.\, (q \,.\, r))$;
T9b: $((p \vee q) \vee r) \equiv (p \vee (q \vee r))$;
T9c: $((p \equiv q) \equiv r) \equiv (p \equiv (q \equiv r))$.

Because of T9a–T9c three-membered conjunctions, alternations, and biconditionals could be written without inner parentheses.

The next four laws are called *laws of Distributivity*. T10a shows that if a conjunction includes an alternation among its factors, then it is logically equivalent to an alternation of conjunctions; T10b shows that if an alternation includes a conjunction among its summands, then it is logically equivalent to a conjunction of alternations; T10c and T10d show that if a conditional has a conjunction or an alternation as its consequent, then it is logically equivalent to a conjunction or to an alternation of conditionals. Many other laws of Distributivity hold which are not listed here.

T10a: $(p \,.\, (q \vee r)) \equiv ((p \,.\, q) \vee (p \,.\, r))$;
T10b: $(p \vee (q \,.\, r)) \equiv ((p \vee q) \,.\, (p \vee r))$;
T10c: $(p \supset (q \,.\, r)) \equiv ((p \supset q) \,.\, (p \supset r))$;
T10d: $(p \supset (q \vee r)) \equiv ((p \supset q) \vee (p \supset r))$.

The following two tautologies are closely related to T10c–T10d; we record them under the same heading: 'Distributivity'.

T10e: $((p \vee q) \supset r) \equiv ((p \supset r) \,.\, (q \supset r))$;
T10f: $((p \,.\, q) \supset r) \equiv ((p \supset r) \vee (q \supset r))$.

T11a and T11b, called *laws of Transitivity*, were known to the Stoics. They are sometimes referred to in the literature as *hypothetical syllogisms* (cf. also T23 and T24a); *categorical syllogisms*, the kernel of Aristotle's *Organon*, will be studied in chapter two.

T11a: $((p \supset q) . (q \supset r)) \supset (p \supset r)$;
T11b: $((p \equiv q) . (q \equiv r)) \supset (p \equiv r)$.

The following is *the law of the Dilemma,* famous in classical rhetoric:

T12: $(((p \supset q) . (r \supset s)) . (p \vee r)) \supset (q \vee s)$.

In view of the associativity of '.', T12 could be written more simply:

$$((p \supset q) . (r \supset s) . (p \vee r)) \supset (q \vee s).$$

The next two laws, known as *laws of Exportation,* show that a conditional with a conjunction as its antecedent is logically equivalent to a conditional with a conditional as its consequent:

T13a: $((p . q) \supset r) \equiv (p \supset (q \supset r))$;
T13b: $((p . q) \supset r) \equiv (q \supset (p \supset r))$.

The next two laws, respectively known as *the law of Factorization* and *the law of Summation,* show that a conditional logically implies the result of conjoining or alternating its components with an arbitrary statement:

T14: $(p \supset q) \supset ((r . p) \supset (r . q))$;
T15: $(p \supset q) \supset ((r \vee p) \supset (r \vee q))$.

The next two laws, known as *laws of Transposition,* show that a conditional or a biconditional is logically equivalent to the result of transposing its components and attaching to each one of them a negation sign:

T16a: $(p \supset q) \equiv (\sim q \supset \sim p)$;
T16b: $(p \equiv q) \equiv (\sim q \equiv \sim p)$.

The next laws state logical equivalences between various molecular formulae. T17a, labelled *Biconditional,* shows that a biconditional is logically equivalent to a conjunction of conditionals:

T17a: $(p \equiv q) \equiv ((p \supset q) . (q \supset p))$.

A weaker version of T17a is appended:

T17b: $(p \equiv q) \supset (p \supset q)$.

T18, labelled *Conditional-Alternation,* shows that a conditional is logically equivalent to an alternation with the denial of its antecedent as first summand and its consequent as second summand:

T18: $(p \supset q) \equiv (\sim p \vee q)$.

T19, labelled *Conditional-Conjunction,* shows that a conditional is logic-

ally equivalent to the denial of a conjunction with its antecedent as first factor and the denial of its consequent as second factor:

T19: $(p \supset q) \equiv \sim(p \,.\, \sim q)$.

T20a and T20b, labelled *Exclusive Alternation*, show that an exclusive alternation is logically equivalent: 1. to the denial of a biconditional, and 2. to an inclusive alternation and the denial of a conjunction. In view of T20b an exclusive alternation may be interpreted as an inclusive alternation whose summands are not jointly true.

T20a: $(p \not\equiv q) \equiv \sim(p \equiv q)$;
T20b: $(p \not\equiv q) \equiv ((p \vee q) \,.\, \sim(p \,.\, q))$.

T21a and T21b, called *laws of Duality*, show that the denial of a conjunction is logically equivalent to an alternation of denials and the denial of an alternation logically equivalent to a conjunction of denials:

T21a: $\sim(p \,.\, q) \equiv (\sim p \vee \sim q)$;
T21b: $\sim(p \vee q) \equiv (\sim p \,.\, \sim q)$.

Under the same heading, 'Duality', are recorded two laws showing that the denial of a biconditional is logically equivalent to a biconditional with a denial as its left-hand or right-hand component:

T21c: $\sim(p \equiv q) \equiv (\sim p \equiv q)$;
T21d: $\sim(p \equiv q) \equiv (p \equiv \sim q)$.

T22a and T22b are labelled *laws of Expansion*; they show that a conditional is logically equivalent to a biconditional with a conjunction or an alternation as its right-hand component:

T22a: $(p \supset q) \equiv (p \equiv (p \,.\, q))$;
T22b: $(p \supset q) \equiv (q \equiv (p \vee q))$.

We end our list with the following four schemata:

T23: $((p \supset q) \,.\, p) \supset q$;
T24a: $((p \supset q) \,.\, \sim q) \supset \sim p$;
T24b: $((p \vee q) \,.\, \sim p) \supset q$;
T24c: $((p \vee q) \,.\, \sim q) \supset p$.

T23 is labelled *Modus Ponens*, T24a–T24c *Modus Tollens*. T23 and T24a were known to the Stoics as *hypothetical syllogisms*; T24b and T24c, as *disjunctive syllogisms*.

Before closing this section we introduce a few metalogical signs.

We appointed above the four Latin letters '*p*', '*q*', '*r*', and '*s*', called

sentential letters, as place-holders for statements, and grouped under the heading 'sentential formulae' both statements and compounds of sentential letters, connectives, and parentheses. Besides using sentential formulae, however, we shall also mention them. We accordingly appoint here a second set of place-holders, the Greek letters 'φ', 'χ', 'ψ', 'ω', and 'τ', the first four as place-holders for names of sentential formulae, the last one as a place-holder for names of sentential truths.[3]

Let us note that:

(*a*) as the four Latin letters '*p*', '*q*', '*r*', and '*s*' may enter as components in sentential schemata, so the five Greek letters 'φ', 'χ', 'ψ', 'ω', and 'τ', may enter as subjects or objects in metaschemata like:

$$\varphi \text{ logically implies } \psi \qquad (9),$$

$$\varphi \text{ is logically equivalent to } \psi \qquad (10),$$

and so on;

(*b*) as the result of replacing every Latin letter in a sentential schema by one of its instances is a statement, so the result of replacing every Greek letter in a metaschema by one of its instances is a metastatement. The result, for example, of respectively replacing the letters 'φ' and 'ψ' in (9) by names of the two statements 'Italy is south of Germany . Germany is south of Sweden' and 'Italy is south of Germany' is a metastatement like:

'Italy is south of Germany . Germany is south of Sweden' logically implies 'Italy is south of Germany';

similarly, the result of respectively replacing in (10) the letters 'φ' and 'ψ' by names of the two statements 'Italy is south of Germany' and '$\sim\sim$(Italy is south of Germany)' is a metastatement like:

'Italy is south of Germany' is logically equivalent to '$\sim\sim$(Italy is south of Germany)'.

To complete our arsenal of place-holders, we adopt two extra metalogical signs: *the left-hand corner* '$\ulcorner$' and *the right-hand corner* '$\urcorner$'. We shall insert '$\ulcorner$' and '$\urcorner$' around sequences:

$$\text{———},$$

of Greek letters, connectives, and parentheses, and use the expressions:

$$\ulcorner\text{———}\urcorner,$$

[3]Subscripts will occasionally be attached to 'φ', 'χ', 'ψ', 'ω', and 'τ' in the following fashion: 'φ_1', 'φ_2', . . . , 'φ_n', 'χ_1', 'χ_2', . . . , 'χ_n', and so on.

as place-holders for the names of the sentential formulae which result from replacing the Greek letters of '——' by sentential formulae or truths.

$$\ulcorner\varphi \,.\, \psi\urcorner \tag{11}$$

for instance, will serve as a place-holder for the names of the conjunctions which result from replacing 'φ' and 'ψ' in '$\varphi \,.\, \psi$' by two sentential formulae;

$$\ulcorner(\varphi \supset \varphi) \equiv \tau\urcorner \tag{12}$$

will serve as a place-holder for the names of the biconditionals which result from respectively replacing 'φ' and 'τ' in '$(\varphi \supset \varphi) \equiv \tau$' by a sentential formula and a sentential truth; and so on.

The two corners '$\ulcorner$' and '$\urcorner$' are idle in

$$\ulcorner\varphi\urcorner,$$

which has indeed the same instances as 'φ'. They are not idle, however, in (11) or (12), as the reader may check by comparing the respective instances of

$$\ulcorner\varphi \,.\, \psi\urcorner$$

and

$$\varphi \,.\, \psi,$$

or the respective instances of

$$\ulcorner(\varphi \supset \varphi) \equiv \tau\urcorner$$

and

$$(\varphi \supset \varphi) \equiv \tau.$$

Let us add that:

(*a*) place-holders of the type:

$$\ulcorner\text{——}\urcorner,$$

may enter as subjects or objects in metaschemata like:

$$\ulcorner\varphi \,.\, \psi\urcorner \text{ logically implies } \ulcorner\varphi \vee \psi\urcorner \tag{13}$$

and

$$\ulcorner\varphi \,.\, \psi\urcorner \text{ is logically equivalent to } \ulcorner\psi \,.\, \varphi\urcorner \tag{14}$$

(*b*) the result of replacing every place-holder of the type:

$$\ulcorner\text{——}\urcorner,$$

in a metaschema by one of its instances is a metastatement. The result, for example, of respectively replacing

$$\ulcorner\varphi \,.\, \psi\urcorner$$

and

$$\ulcorner \varphi \vee \psi \urcorner$$

in (13) by names of the two statements 'John is sick . Mary is away' and 'John is sick v Mary is away' is a metastatement like:

'John is sick . Mary is away' logically implies 'John is sick v Mary is away';

similarly, the result of respectively replacing

$$\ulcorner \varphi \, . \, \psi \urcorner$$

and

$$\ulcorner \psi \, . \, \varphi \urcorner$$

in (14) by names of the two statements 'John is sick . Mary is away' and 'Mary is away . John is sick' is a metastatement like:

'John is sick . Mary is away' is logically equivalent to 'Mary is away . John is sick'.[4]

8. RULES OF SENTENTIAL DEDUCTION

Deduction may be defined as the process whereby a given set of statements $\varphi_1, \varphi_2, \ldots, \varphi_n$ $(n \geq 1)$, called *premises*, is offered as logical evidence for asserting another statement ψ, called its *conclusion.*

The concept of deduction is closely related to the concept of logical implication. It is clear that if a set of statements $\varphi_1, \varphi_2, \ldots, \varphi_n$ logically implies another statement ψ, and $\varphi_1, \varphi_2, \ldots, \varphi_n$ are assumed as premises, then ψ may be deduced as a conclusion from $\varphi_1, \varphi_2. \ldots, \varphi_n$. It is clear, conversely, that if a given statement ψ is to be deduced as a conclusion from a set of statements $\varphi_1, \varphi_2, \ldots, \varphi_n$ assumed as premises, then $\varphi_1, \varphi_2, \ldots, \varphi_n$ must logically imply ψ. A given statement ψ is this deducible as a conclusion from a set of premises $\varphi_1, \varphi_2, \ldots, \varphi_n$ if and only if $\varphi_1, \varphi_2, \ldots, \varphi_n$ logically imply ψ.

Our present task is to devise rules for carrying out deductions. We shall devise four such rules, respectively numbered R1, R2, R3, and R4. R1–R4 will be adequate for all deductive purposes in sentential logic. It can indeed be shown that:

(*a*) if a conclusion ψ is deduced from a set of premises $\varphi_1, \varphi_2, \ldots, \varphi_n$ in accordance with R1, R2, R3, and/or R4, then

$$\ulcorner(\varphi_1 \, . \, \varphi_2 \, . \, \ldots \, . \, \varphi_n) \supset \psi \urcorner$$

[4]The use of corners in metalogic started with W. V. Quine; for further details, see his *Mathematical Logic*, pp. 33–37. Several logicians who dispense with corners adopt equivalent conventions.

is sententially true, and hence the set of premises $\varphi_1, \varphi_2, \ldots, \varphi_n$ sententially implies ψ;

and that:

(*b*) if

$$\ulcorner(\varphi_1 \,.\, \varphi_2 \,.\, \ldots \,.\, \varphi_n) \supset \psi\urcorner$$

is sententially true and hence the set of premises $\varphi_1, \varphi_2, \ldots, \varphi_n$ sententially implies ψ, then ψ can be deduced from $\varphi_1, \varphi_2, \ldots, \varphi_n$ in accordance with R1, R2, R3, and/or R4.

Our first rule of deduction is called *Detachment*; it reads: "From φ and $\ulcorner\varphi \supset \psi\urcorner$ one may deduce ψ (R1)." The rule of Detachment enables us to break a conditional and assert its consequent as a conclusion whenever the conditional in question and its antecedent are given as premises or, as the case may be, deducible from a set of premises. We append, for illustration's sake, sample uses of Detachment. 1. Given the two premises 'John is elected' and 'John is elected ⊃ Mary is promoted', one may deduce from them the conclusion 'Harry is promoted'. 2. Given the three premises 'John is elected', 'John is elected ⊃ Harry is promoted', and 'Harry is promoted ⊃ Jerry is demoted', one may deduce the initial conclusion 'Harry is promoted' from the first two premises and deduce the final conclusion 'Jerry is demoted' from 'Harry is promoted' and the third premise.

R1 is readily justified since by T23 $\ulcorner(\varphi \supset \psi) \,.\, \varphi\urcorner$ logically implies ψ. Beware, however, of confusing Detachment and *Modus Ponens*; the former is a metalogical recipe for drawing conclusions from premises; the latter is a logical schema which yields sentential truths whenever statements are substituted for its letters.[5]

The second rule, called *Adjunction*, reads: "From φ and ψ one may deduce $\ulcorner\varphi \,.\, \psi\urcorner$ (R2)." To see how the rule of Adjunction works in practice, let us assume the following three premises:

P1: (John is an American . John lives in the United States) ⊃ John drinks Coca-Cola by the barrel,
P2: John is an American,
P3: John lives in the United States,

and attempt to deduce from them the conclusion: 'John drinks Coca-Cola by the barrel'. According to the rule of Detachment, this conclusion should follow from P1 and

John is an American . John lives in the United States (1).

[5]In several treatises on logic R1 is called *Modus Ponens*; note that the distinction between R1 and T23 is not thereby obliterated.

We have not assumed (1), however; we have assumed instead the two premises P2 and P3. If we are to reach our conclusion, we must therefore link P2 and P3 by a '.' sign. The rule of Adjunction allows us to take such a step; it is readily justified since by T3a $\ulcorner\varphi\ .\ \psi\urcorner$ logically implies itself.

Let us now embark upon more complex deductions. The conditional:

((Napoleon proved unequal to himself at Waterloo $\vee$ Napoleon was crushed by fate at Waterloo) . $\sim$(Napoleon proved unequal to himself at Waterloo)) $\supset$ Napoleon was crushed by fate at Waterloo,

is sententially true, it being an instance of a valid schema, T24b. We may thus conclude that the two premises:

P1: Napoleon proved unequal to himself at Waterloo $\vee$ Napoleon was crushed by fate at Waterloo (2),

P2: $\sim$(Napoleon proved unequal to himself at Waterloo) (3),

sententially imply:

Napoleon was crushed by fate at Waterloo (4).

Yet we cannot deduce (4) from (2) and (3) with the above rules. Adjunction will lead us from (2) and (3) to

(2) . (3).[6]

This, however, is as close as we can get to our conclusion (4). We apparently need at this point a new rule allowing for the deduction of ψ from $\ulcorner(\varphi \vee \psi)\ .\ \sim\varphi\urcorner$. Such a rule, let us number it R3′, would do the trick here; it would fail us, however, in other cases.

The conditional:

((I read James Joyce $\supset$ I get a headache) . (I get a headache $\supset$ I must take an aspirin)) $\supset$ (I read James Joyce $\supset$ I must take an aspirin),

is sententially true, it being an instance of a valid schema, T11a. We may thus conclude that the two premises:

P1: I read James Joyce $\supset$ I get a headache (5),

P2: I get a headache $\supset$ I must take an aspirin (6),

sententially imply:

I read James Joyce $\supset$ I must take an aspirin (7).

[6]We take from now on the liberty of using '(2)', '(3)', and so on, as abbreviations for statements whenever '(2)', '(3)', and so on, occur in such displayed contexts as:

(2) . (3),
((2) . (3)) $\supset$ (2),

and so on; otherwise, of course, '(2)', '(3)', and so on, serve as names of statements.

Yet we cannot deduce (7) from (5) and (6) with the sole help of R1, R2, and R3′. Adjunction again will lead us from (5) and (6) to

$$(5) \,.\, (6).$$

This, however, is as close as we can get to our conclusion (7). We apparently need at this point a new rule, say R3″, allowing for the deduction of $\ulcorner\varphi \supset \chi\urcorner$ from $\ulcorner(\varphi \supset \psi) \,.\, (\psi \supset \chi)\urcorner$.

From these two examples one might gather that Detachment and Adjunction must be supplemented with an infinite number of rules if all sentential deductions are to be carried out. There is no need, fortunately, for such a wasteful measure; the same result can be achieved through a single rule. Whenever a conclusion ψ, though sententially implied by a set of premises $\varphi_1, \varphi_2, \ldots, \varphi_n$, is not deducible from them through Detachment and/or Adjunction, one can spot a set of sentential truths $\tau_1, \tau_2, \ldots, \tau_p$ ($p \geq 1$) such that ψ is deducible from the enlarged set of premises $\varphi_1, \varphi_2, \ldots, \varphi_n, \tau_1, \tau_2, \ldots, \tau_p$, through Detachment and/or Adjunction. On meeting such a recalcitrant conclusion ψ, one may therefore supplement its premises with appropriate τ's and resume the proof of ψ along familiar lines.

For example, one can deduce (7) from (5) and (6) through Detachment and Adjunction if one assumes the following sentential truth:

((I read James Joyce ⊃ I get a headache) . (I get a headache ⊃ I must take an aspirin)) ⊃ (I read James Joyce ⊃ I must take an aspirin) (8).

The deduction now proceeds as follows:

1. From (5) and (6) one deduces:

$$(5) \,.\, (6) \qquad (9),$$

by the rule of Adjunction;

2. From (8) and (9) one deduces (7) by the rule of Detachment.

Similarly one can deduce (4) from (2) and (3) through Detachment and Adjunction if one assumes the following sentential truth:

((Napoleon proved unequal to himself at Waterloo ∨ Napoleon was crushed by fate at Waterloo) . ∼(Napoleon proved unequal to himself at Waterloo)) ⊃ Napoleon was crushed by fate at Waterloo (10).

The deduction now proceeds as follows:

1. From (2) and (3) one deduces:

$$(2) \,.\, (3) \qquad (11),$$

by the rule of Adjunction;

2. From (10) and (11) one deduces (4) by the rule of Detachment.

The present device is legitimate, for a set of premises $\varphi_1, \varphi_2, \ldots, \varphi_n$ will sententially imply a conclusion ψ if and only if the enlarged set of premises $\varphi_1, \varphi_2, \ldots, \varphi_n, \tau_1, \tau_2, \ldots, \tau_p$ itself does. Being legitimate, it may be raised to the status of a rule, *the rule of Insertion*, tentatively phrased as follows: "If τ is an instance of a valid sentential schema, then τ may serve as a premise in any sentential deduction."

There is an infinite number of valid schemata. Some are fairly obvious and simple in structure; we recorded them in section 7. Others are less obvious and too complex in structure to be worth recording. As phrased above, the rule of Insertion covers both types of schemata; we weaken it to read: "If τ is an instance of one of schemata T1–T24c, then τ may serve as a premise in any sentential deduction;" or, more simply: "If τ is an instance of a recorded schema, then τ may serve as a premise in any sentential deduction (R3)."

The application of this rule calls for a few comments.

(1) When a set of premises $\varphi_1, \varphi_2, \ldots, \varphi_n$ sententially implies a conclusion ψ and the conditional:

$$\ulcorner(\varphi_1 \,.\, \varphi_2 \,.\, \ldots \,.\, \varphi_n) \supset \psi\urcorner \qquad (12),$$

is an instance of a recorded schema, the indicated step is to assume (12) as a further premise. ψ will then follow from $\varphi_1, \varphi_2, \ldots, \varphi_n$ and (12) through Detachment and, possibly, Adjunction. It is along such lines that we deduced (7) from (5), (6), and (8) above; we relied upon the fact that the schema:

$$((p \supset q) \,.\, (q \supset r)) \supset (p \supset r),$$

of which (8) is an instance, is recorded in section 7.

(2) When, however, (12) is not an instance of a recorded schema, then one has to spot instances $\tau_1, \tau_2, \ldots, \tau_p$ of recorded schemata such that $\varphi_1, \varphi_2, \ldots, \varphi_n, \tau_1, \tau_2, \ldots, \tau_p$ yield ψ through Detachment and/or Adjunction. The relevant τ's can always be detected, even if their detection may sometimes tax the ingenuity of the beginner.

To illustrate the above remark, let us carry out one deduction falling under case (2); another deduction of the same type will be carried out in section 9. Let us assume the following two premises:

P1: Franco is overthrown $\supset$ Spain will become a communist stronghold,
P2: $\sim$(Franco is overthrown) $\supset$ Spain will become a fascist stronghold,

and deduce from them the conclusion:

Spain will become a communist stronghold $\vee$ Spain will become a fascist stronghold.

To simplify matters, we may abbreviate 'Franco is overthrown' as 'O', 'Spain will become a communist stronghold' as 'C', and 'Spain will become a fascist stronghold' as 'F'. We are thus given:

P1: $O \supset C$,
P2: $\sim O \supset F$,

and asked to deduce:

$$C \vee F.$$

The conditional:

$$((O \supset C) . (\sim O \supset F)) \supset (C \vee F),$$

is an instance of a valid schema; it is not, however, an instance of a recorded one. We must therefore spot instances of recorded schemata which will allow for the deduction of '$C \vee F$' from '$O \supset C$' and '$\sim O \supset F$' through Detachment and/or Adjunction. The deduction may run as follows:

1. We assume two sentential truths:

$$O \vee \sim O \qquad (13)$$

and

$$(((O \supset C) . (\sim O \supset F)) . (O \vee \sim O)) \supset (C \vee F) \qquad (14);$$

these two assumptions are permissible, (13) being an instance of the recorded schema T1 and (14) an instance of the recorded schema T12.

2. We deduce '$(O \supset C) . (\sim O \supset F)$' from the two premises '$O \supset C$' and '$\sim O \supset F$' by using the rule of Adjunction.

3. We deduce '$((O \supset C) . (\sim O \supset F)) . (O \vee \sim O)$' from step two and (13) by using again the rule of Adjunction.

4. We detach '$C \vee F$' from (14) by using the rule of Detachment whose provisos are satisfied by step three and (14).

Since R3 authorizes the insertion of τ's in a deduction and τ's are substitution instances of schemata, a word on substitution is in order. We shall encounter in this book many types of substitution; for the time being, however, we may restrict ourselves to one: *sentential substitution.* Sentential substitution is the process whereby statements are substituted for the letters of a sentential formula. We perform, for illustration's sake, the substitutions leading from T12:

$$(((p \supset q) . (r \supset s)) . (p \vee r)) \supset (q \vee s) \qquad (15),$$

to (14):

$$(((O \supset C) . (\sim O \supset F)) . (O \vee \sim O)) \supset (C \vee F).$$

1. We first substitute 'O' for 'p' in (15) to obtain:

$$(((O \supset q) . (r \supset s)) . (O \vee r)) \supset (q \vee s) \qquad (16);$$

2. we next substitute 'C' for 'q' in (16) to obtain:

$$(((O \supset C) \,.\, (r \supset s)) \,.\, (O \vee r)) \supset (C \vee s) \qquad (17);$$

3. we next substitute '$\sim$O' for 'r' in (17) to obtain:

$$(((O \supset C) \,.\, ({\sim}O \supset s)) \,.\, (O \vee {\sim}O)) \supset (C \vee s) \qquad (18);$$

4. we finally substitute 'F' for 's' in (18) to obtain (14).

The following three points should be borne in mind:

1. A given statement may, though of course it need not, be substituted for more than one letter in a sentential formula. For example, both the 'p' and the 'q' of T6a:

$$(p \,.\, q) \supset p,$$

may be replaced by the same statement 'John is sick' to yield:

(John is sick . John is sick) $\supset$ John is sick.

2. If a statement is substituted for one occurrence of a letter in a formula, then the same statement must be substituted for all the occurrences of that letter in the formula. For example, if 'John is sick' is substituted for the first occurrence of 'p' in

$$(p \,.\, q) \supset p,$$

then 'John is sick' must be substituted for the second occurrence of 'p' in that formula; note indeed that if 'John is sick' were substituted for the first occurrence of 'p', 'Paul is mad' for 'q', and '$\sim$(John is sick)' for the second occurrence of 'p', then the valid schema:

$$(p \,.\, q) \supset p,$$

would yield:

(John is sick . Paul is mad) $\supset$ $\sim$(John is sick),

a statement which, clearly, is not sententially true.

3. One may substitute both atomic and molecular statements for the letters of a given formula, but one cannot substitute statements, whether atomic or molecular, for its molecular components. For instance, one cannot substitute the atomic component 'John is sick' nor even the molecular statement 'John is sick . Mary will come' for the molecular component '$p \,.\, q$' of

$$(p \,.\, q) \supset p.$$

Note indeed that if such a substitution were performed and any other

statement than 'John is sick' substituted for the second occurrence of 'p', then the valid schema:

$$(p \,.\, q) \supset p,$$

would yield statements like:

$$\text{John is sick} \supset \sim(\text{John is sick}),$$

$$(\text{John is sick} \,.\, \text{Mary will come}) \supset \sim(\text{Mary will come}),$$

and so on, which, clearly, are not sententially true.

We have met so far three rules of sentential deduction: Detachment, Adjunction, and Insertion. They may be summed up as follows:

R1: "From φ and $\ulcorner\varphi \supset \psi\urcorner$ one may deduce ψ."
R2: "From φ and ψ one may deduce $\ulcorner\varphi \,.\, \psi\urcorner$."
R3: "If τ is an instance of a recorded schema, then τ may serve as a premise in any sentential deduction."

With R1 and R3 at hand we might dispense with R2, as we shall establish in section 28; we retain it, though, to shorten deductions. For the same purpose we introduce a fourth rule, also dispensable, *the rule of Interchange.*

Let us assume that we are given the following two premises:

P1: $O \supset C$,
P2: $C \equiv \sim F$,

where 'O' stands for 'Franco is overthrown', 'C' for 'Spain will become a communist stronghold', and 'F' for 'Spain will become a fascist stronghold', and asked to deduce from them the conclusion:

$$O \supset \sim F.$$

We might insert '$(C \equiv \sim F) \supset (C \supset \sim F)$', deduce '$C \supset \sim F$' from '$(C \equiv \sim F) \supset (C \supset \sim F)$' and P2, deduce '$(O \supset C) \,.\, (C \supset \sim F)$' from P1 and '$C \supset \sim F$', insert '$((O \supset C) \,.\, (C \supset \sim F)) \supset (O \supset \sim F)$', and deduce '$O \supset \sim F$' from '$(O \supset C) \,.\, (C \supset \sim F)$' and '$((O \supset C) \,.\, (C \supset \sim F)) \supset (O \supset \sim F)$', thus reaching '$O \supset \sim F$' in five steps. We might also, in view of biconditional P2, interchange 'C' and '$\sim F$' in P1, thus reaching '$O \supset \sim F$' in one step. Such a shortcut is now authorized by the rule of Interchange, which reads: 'From φ and $\ulcorner\psi \equiv \chi\urcorner$ one may deduce ω, if ω is the result of interchanging ψ and χ at one or more places in φ (R4)."

This rule allows for two different types of interchange, depending on whether ψ or χ is only part of or the whole of φ. The first possibility materialized above when we interchanged 'C' and '$\sim F$' in '$O \supset C$'. To exemplify the second possibility, let us again assume '$O \supset C$' and '$C \equiv$

$\sim$F' as premises and deduce from them the conclusion 'O $\equiv$ (O . $\sim$F)'. We may first interchange 'C' and '$\sim$F' in 'O $\supset$ C' to obtain 'O $\supset$ $\sim$F'; we may next insert the following biconditional:

$$(\mathrm{O} \supset \sim\mathrm{F}) \equiv (\mathrm{O} \equiv (\mathrm{O} \,.\, \sim\mathrm{F})),$$

(an instance of T22a), and we may finally interchange 'O $\supset$ $\sim$F' and 'O $\equiv$ (O . $\sim$F)' in 'O $\supset$ $\sim$F' itself to obtain 'O $\equiv$ (O . $\sim$F)', the desired conclusion. In the last interchange, the reader will note, ψ was the whole of φ.

In both applications of the rule of Interchange, ψ or χ occurred in φ only once; if, however, ψ or χ had occurred n times ($n > 1$) in φ, then we could have interchanged ψ and χ in φ m times ($m \leq n$), had we chosen to. This is one important aspect in which interchange differs from substitution. Given 'O $\equiv$ (O . $\sim$F)' and 'O $\equiv$ $\sim\sim$O' (cf. T5), for example, we could deduce from them by R4 '$\sim\sim$O $\equiv$ (O . $\sim$F)' or 'O $\equiv$ ($\sim\sim$O . $\sim$F)' as well as '$\sim\sim$O $\equiv$ ($\sim\sim$O . $\sim$F)'.

9. SAMPLE DEDUCTIONS

We have seen above that three factors may be at play in a sentential deduction: statements assumed as premises, τ's enlarging the initial stock of premises, and rules endorsing each step of the deduction. With these elements at hand we may formalize the concept of a *sentential deduction.* A sentential deduction is a series of statements, called *steps,* each member of which is either a premise or an instance τ of a recorded schema or follows from previous members of the series through Detachment, Adjunction, or Interchange; the last member of a sentential deduction is what we called above a *conclusion.*

We adopt three conventions for recording sentential deductions:

1. Each statement entering a deduction will be written on a separate line and numbered, except for the conclusion which is preceded by the two letters 'Cl'; premises and statements assumed through R3 will also be preceded by the letter 'P'.

2. Substitutions performed in a recorded schema φ to obtain an instance τ of φ will be quoted to the right of τ along with the letters 'Ins' and the reference number of φ. The following may serve as a sample:

Pn: O $\vee$ $\sim$O   Ins T1;O/p.

3. After each step of a deduction will be quoted both the statements from which and the rule through which the step follows. The rules of Detachment, Adjunction, and Interchange may respectively be referred to as Det, Adj, and Int.

To show these conventions at work we shall formalize some of the deductions sketched in the previous section.

Example 1: On page 24 we were given the following two premises:

P1: Napoleon proved unequal to himself at Waterloo $\vee$ Napoleon was crushed by fate at Waterloo,
P2: $\sim$(Napoleon proved unequal to himself at Waterloo),

and asked to deduce from them the conclusion:

Cl: Napoleon was crushed by fate at Waterloo.

Using 'U' as an abbreviation for 'Napoleon proved unequal to himself at Waterloo' and 'F' as an abbreviation for 'Napoleon was crushed by fate at Waterloo', we may adopt the following deduction:

P1:	$U \vee F$	
P2:	$\sim U$	
P3:	$((U \vee F) . \sim U) \supset F$	Ins T24b; U/p, F/q
4:	$(U \vee F) . \sim U$	Adj 1 and 2
Cl:	F	Det 3 and 4

Example 2: On page 24 we were given the following two premises:

P1: I read James Joyce $\supset$ I get a headache,
P2: I get a headache $\supset$ I must take an aspirin,

and asked to deduce from them the conclusion:

Cl: I read James Joyce $\supset$ I must take an aspirin.

Using 'J' as an abbreviation for 'I read James Joyce', 'H' as an abbreviation for 'I get a headache', and 'A' as an abbreviation for 'I must take an aspirin', we may adopt the following deduction:

P1:	$J \supset H$	
P2:	$H \supset A$	
P3:	$((J \supset H) . (H \supset A)) \supset (J \supset A)$	Ins T11a; J/p, H/q, A/r
4:	$(J \supset H) . (H \supset A)$	Adj 1 and 2
Cl:	$J \supset A$	Det 3 and 4

Example 3: On page 26 we were given the two premises: '$O \supset C$' and '$\sim O \supset F$', and asked to deduce from them the conclusion: '$C \vee F$'. The following deduction may be adopted:

P1:	$O \supset C$	
P2:	$\sim O \supset F$	
P3:	$O \vee \sim O$	Ins T1; O/p

P4:	$(((O \supset C) . (\sim O \supset F)) . (O \vee \sim O)) \supset (C \vee F)$	Ins T12; O/p, C/q, $\sim O/r$, F/s
5:	$(O \supset C) . (\sim O \supset F)$	Adj 1 and 2
6:	$((O \supset C) . (\sim O \supset F)) . (O \vee \sim O)$	Adj 5 and 3
Cl:	$C \vee F$	Det 4 and 6

Example 4: On page 29 we were given the two premises: '$O \supset C$' and '$C \equiv \sim F$', and asked to deduce from them the conclusion: '$O \supset \sim F$'. Two deductions were sketched. The first one ran as follows:

P1:	$O \supset C$	
P2:	$C \equiv \sim F$	
P3:	$(C \equiv \sim F) \supset (C \supset \sim F)$	Ins T17b; C/p, $\sim F/q$
P4:	$((O \supset C) . (C \supset \sim F)) \supset (O \supset \sim F)$	Ins T11a; O/p, C/q, $\sim F/r$
5:	$C \supset \sim F$	Det 2 and 3
6:	$(O \supset C) . (C \supset \sim F)$	Adj 1 and 5
Cl:	$O \supset \sim F$	Det 4 and 6

The second deduction, using R4, had only three lines:

P1:	$O \supset C$	
P2:	$C \equiv \sim F$	
Cl:	$O \supset \sim F$	Int 1 and 2

Example 5: Let us assume that we are given the two premises:

P1: Inflation comes if salaries and prices rise,
P2: Prices rise if salaries do,

and asked to deduce from them the conclusion:

Cl: Inflation comes if salaries rise.

The following deduction may be adopted, where 'S' stands for 'Salaries rise', 'P' for 'Prices rise', and 'I' for 'Inflation comes':

P1:	$(S . P) \supset I$	
P2:	$S \supset P$	
P3:	$((S . P) \supset I) \equiv (P \supset (S \supset I))$	Ins T13b; S/p, P/p, I/r
P4:	$((S \supset P) . (P \supset (S \supset I))) \supset (S \supset (S \supset I))$	Ins T11a; S/p, P/q, $S \supset I/r$
P5:	$((S . S) \supset I) \equiv (S \supset (S \supset I))$	Ins T13a; S/p, S/q, I/r
P6:	$(S . S) \equiv S$	Ins T4a; S/p
7:	$P \supset (S \supset I)$	Int 1 and 3
8:	$(S \supset P) . (P \supset (S \supset I))$	Adj 2 and 7
9:	$S \supset (S \supset I)$	Det 4 and 8
10:	$(S . S) \supset I$	Int 5 and 9
Cl:	$S \supset I$	Int 6 and 10

10. THE TRUTH-TABLE METHOD

There is a mechanical procedure for identifying sentential truths and sententially valid schemata: *the truth-table method.* It is based on two principles. The first one reads:

P1: Every statement is either true or false,

or, to use a more technical phrasing:

P1′: Every statement has one of the two truth-values: 'is true' (abbreviated 'T') or 'is false' (abbreviated 'F').

The second one reads:

P2: The truth-values of a sentential compound are determined by the truth-values of its components.

To give a more graphic form to P1′, we may write 'T' and 'F' under an unspecified statement φ as in:

Figure one

φ
T
F

Figure one, called a *reference table* or *reference column,* serves to compute the truth-values of our first sentential compound: $\ulcorner \sim\varphi \urcorner$. Two cases arise:

Case 1: φ is true;
Case 2: φ is false.

We agreed above (page 10) to consider $\ulcorner \sim\varphi \urcorner$ false in the first case, true in the second. This decision is recorded in the following table, called *the defining table* or *defining matrix of '$\sim$'*:

Figure two

φ	$\ulcorner \sim\varphi \urcorner$
T	F
F	T

When turning to $\ulcorner \varphi \,.\, \psi \urcorner$, $\ulcorner \varphi \vee \psi \urcorner$, $\ulcorner \varphi \supset \psi \urcorner$, $\ulcorner \varphi \equiv \psi \urcorner$, and $\ulcorner \varphi \not\equiv \psi \urcorner$, we must enlarge our reference table to cover four possibilities. The resulting table (figure three) records the fact that, given two statements:

1. both are true,
or 2. the first one is false and the second one is true,

or 3. the first one is true and the second one is false,
or 4. both are false.

Figure three

φ	ψ
T	T
F	T
T	F
F	F

Figure three may be built after figure one. We first enter under φ the two truth-values 'T' and 'F' and assume that ψ is true in both cases; we then reenter under φ the two truth-values 'T' and 'F' and assume that ψ is false in both cases. This operation involves four steps:

1.	φ	ψ	2.	φ	ψ	3.	φ	ψ	4.	φ	ψ
	T			T	T		T	T		T	T
	F			F	T		F	T		F	T
							T			T	F
							F			F	F

We thus alternate 'T' and 'F' one by one in the reference column for φ, two by two in the reference column for ψ.

With figure three at hand, we can set up defining matrices for all our binary connectives (cf. pp. 10–12):

Figure four

φ	ψ	$\ulcorner \varphi \,.\, \psi \urcorner$	$\ulcorner \varphi \vee \psi \urcorner$	$\ulcorner \varphi \supset \psi \urcorner$	$\ulcorner \varphi \equiv \psi \urcorner$	$\ulcorner \varphi \not\equiv \psi \urcorner$
T	T	T	T	T	T	F
F	T	F	T	T	F	T
T	F	F	T	F	F	T
F	F	F	F	T	T	F

The truth-table method serves to compute the truth-values of sentential schemata and molecular statements; to simplify matters, we shall restrict ourselves here to sentential schemata. The method works as follows:

1. The schema to be tested is written down.
2. Its letters are listed on the left in the order of their appearance in the schema.
3. Under each letter is set up a reference column. These reference columns have two rows when the schema contains one letter; four rows

when the schema contains two letters; eight rows when the schema contains three letters; and, more generally, 2^n rows when the schema contains n letters. In the first reference column, 'T' and 'F' are alternated one by one; in the second, two by two; in the third, four by four; and, more generally, in the nth column, 2^{n-1} by 2^{n-1}.

Let us illustrate these initial steps with respect to three schemata. The first one is '$\sim(p \,.\, \sim p)$', which contains only one letter; we get:

p	$\sim(p \,.\, \sim p)$
T	
F	

The second is '$(p \,.\, (q \supset p)) \supset q$', which contains two letters; we get:

p	q	$(p \,.\, (q \supset p)) \supset q$
T	T	
F	T	
T	F	
F	F	

The third one is '$\sim(((p \supset q) \,.\, (q \supset r)) \supset (p \supset r))$', which contains three letters; we get:

p	q	r	$\sim(((p \supset q) \,.\, (q \supset r)) \supset (p \supset r))$
T	T	T	
F	T	T	
T	F	T	
F	F	T	
T	T	F	
F	T	F	
T	F	F	
F	F	F	

We may now proceed with the final steps of our test. Every schema has a major connective and, possibly, minor ones. We first take up each minor connective, compute the truth-values of the compound built around it, and enter the resulting 'T's and 'F's under the connective. To compute the truth-values in question we must look up the defining value of the connective and the truth-values of the formulae which the connective governs; the latter truth-values will be found in the reference columns if the formulae happen to be letters, under the major connectives of the formulae if they happen to be schemata. Turning finally to the

major connective, we compute and record in like manner the truth-values of the entire schema.

To illustrate this procedure let us work out the above examples. We shall insert numerals under each column to indicate the order of the operations.

Example 1:

p	$\sim$	(p .	$\sim p$)
T	T	F	F
F	T	F	T
	3	2	1

This schema is a denial, the denial of '$p \,.\, \sim p$'. To compute its truth-values we must therefore compute the truth-values of '$p \,.\, \sim p$'; and to compute the truth-values of '$p \,.\, \sim p$', compute the truth-values of '$\sim p$'; the truth-values of 'p' are already known from the reference column. These three operations are easily performed by consulting the defining matrices of '$\sim$' and '.'.

Example 2:

p	q	(p .	($q \supset p$))	$\supset q$
T	T	T	T	T
F	T	F	F	F
T	F	T	T	T
F	F	F	T	T
		2	1	3

This schema is a conditional; to compute its truth-values we must therefore compute the truth-values of '$p \,.\, (q \supset p)$' and 'q'. The truth-values of 'q' are found in the second reference column; those of '$p \,.\, (q \supset p)$' must be computed from the truth-values of 'p' and '$q \supset p$'. The truth-values of 'p' are found in the first reference column; those of '$q \supset p$' must be computed from truth-values of 'p' and 'q'.

Example 3:

p	q	r	$\sim$	((($p \supset q$)	.	($q \supset r$))	$\supset$	($p \supset r$)
T	T	T	F	T	T	T	T	T
T	F	T	F	T	T	T	T	T
F	T	T	F	F	F	F	T	T
F	F	T	F	T	T	T	T	T
T	T	F	F	T	F	F	T	F
F	T	F	F	T	F	F	T	T
T	F	F	F	F	F	T	T	F
F	F	F	F	T	T	T	T	T
			6	1	4	2	5	3

In this last example it obviously does not matter in which order operations 1, 2, and 3 are performed. 1 and 2 must, however, be performed before 4; 3 and 4 must be performed before 5; and 5 must be performed before 6.

When the test is over, the table under the main connective may:

1. consist only of 'T's (cf. example 1),
or 2. consist both of 'T's and 'F's (cf. example 2),
or 3. consist only of 'F's (cf. example 3).

In the first case the schema tested is sometimes called a *tautology*; in the third case, a *contradiction*.

We remarked above that a sentential schema may:

1. yield only truths whatever statements be substituted for its letters,
or 2. yield both truths and falsehoods depending on the statements substituted for its letters,
or 3. yield only falsehoods whatever statements be substituted for its letters.

We called schemata of type one *sententially valid schemata*; we shall respectively call schemata of type two and of type three *sententially indeterminate* and *sententially contravalid schemata.*

It is clear that if a schema φ, after being subjected to the truth-table test, shows only 'T's under its main connective, then φ admits only true statements as its instances and hence is sententially valid; it is clear also that if a schema φ, after being subjected to the truth-table test, shows both 'T's and 'F's under its main connective, then φ admits both true and false statements as its instances and hence is sententially indeterminate; it is clear finally that if a schema φ, after being subjected to the truth-table test, shows only 'F's under its main connective, then φ admits only false statements as its instances and hence is sententially contravalid. The truth-table test may accordingly serve as a test of sentential validity, indeterminacy, and contravalidity.

The truth-table test, applied so far to sentential schemata, may also be applied to statements. Three possibilities may again materialize:

1. the statement is assigned only 'T's, in which case it is sententially and hence logically true;
or 2. it is assigned only 'F's, in which case it is sententially and hence logically false;
or 3. it is assigned both 'T's and 'F's.

In the third case the statement tested may either be true or be false; if it is true, then it is either logically true like:

$$\text{Everything is black} \supset \text{Something is black,}$$

or factually true like:

(New York is north of Trenton . Boston is north of New York) ⊃ Boston is north of Trenton;

if it is logically true, then it may be what we called above *quantificationally true*. The truth-table test is thus only a partial test of logical truth; we shall see in chapter two that no test of quantificational truth and hence no complete test of logical truth can be devised.

We may next show the bearing of our test on sentential deductions. R3, the rule of Insertion, read: "If τ is an instance of a *recorded* valid schema, then τ may serve as a premise in any sentential deduction." Now that we have at hand a mechanical way of identifying valid schemata, we might lift the italicized restriction and phrase R3 as follows: "If τ is an instance of a valid schema, then τ may serve as a premise in any sentential deduction;" or, alternatively: "If τ is sententially true, then τ may serve as a premise in any sentential deduction." With this new rule sentential deductions could be so shortened as to become trivial. Whenever a set of premises $\varphi_1, \varphi_2, \ldots \varphi_n$ sententially implies a conclusion ψ, we could assume the conditional:

$$\ulcorner(\varphi_1 \,.\, \varphi_2 \,.\, \ldots \,.\, \varphi_n) \supset \psi\urcorner,$$

as a further premise, whether it be an instance of a recorded schema or not, and deduce ψ in a step or two. This method of deduction has obvious advantages over the one presented in sections 8 and 9. We do not adopt it as our official method, though, because it works only within sentential logic. In quantificational logic, for example, we shall use logical premises which cannot be mechanically tested and hence need the techniques learned above; we accordingly prefer a unique and comprehensive method of deduction for the whole of logic.

Besides identifying sentential truths and cutting down sentential deductions, the truth-table method plays other important roles. First it may serve as a test of sentential deducibility. We agreed above that a conclusion ψ is sententially deducible from a set of premises $\varphi_1, \varphi_2, \ldots \varphi_n$ if and only if the conditional:

$$\ulcorner(\varphi_1 \,.\, \varphi_2 \,.\, \ldots \,.\, \varphi_n) \supset \psi\urcorner,$$

is sententially true. We may thus use our test of sentential truth as a test of sentential deducibility and determine whether a given conclusion ψ is sententially deducible from a set of premises $\varphi_1, \varphi_2, \ldots, \varphi_n$ without embarking upon a deduction of ψ; we need only form a conjunction out of $\varphi_1, \varphi_2, \ldots, \varphi_n$, insert a '⊃' sign between that conjunction and ψ, and compute the truth-values of the ensuing conditional. Once the truth-table test has been carried out, three possibilities may materialize:

1. a solid column of 'T's stands under the '$\supset$' sign, in which case the conclusion is sententially and hence logically deducible from the premises; or 2. a solid column of 'F's stands under the '$\supset$' sign, in which case the conclusion is not logically deducible from the premises;
or 3. both 'T's and 'F's stand under the '$\supset$' sign, in which case the conclusion is not sententially deducible from the premises, though it may still be quantificationally deducible from them.

Only the first two results are thus final. The second is of special note: if

$$\ulcorner(\varphi \,.\, \varphi_2 \,.\, \ldots \,.\, \varphi_n) \supset \psi\urcorner$$

is a contradiction, then ψ is neither sententially nor quantificationally deducible from $\varphi_1, \varphi_2, \ldots, \varphi_n$.

Finally the truth-table method may serve as an inconsistency test. A set of formulae $\varphi_1, \varphi_2, \ldots, \varphi_n$ is said to be *logically inconsistent* when the n-factored conjunction:

$$\ulcorner\varphi_1 \,.\, \varphi_2 \,.\, \ldots \,.\, \varphi_n\urcorner,$$

is logically false. Given a set of formulae $\varphi_1, \varphi_2, \ldots, \varphi_n$, we may thus test its inconsistency by inserting '.' signs between $\varphi_1, \varphi_2, \ldots, \varphi_n$ and computing the truth-values of the ensuing conjunction. If the conjunction is assigned only 'F's, then the original set of formulae is sententially and hence logically inconsistent; if, on the other hand, the conjunction is assigned at least one 'T', then the original set of formulae may still be inconsistent, but its inconsistency is not sentential and hence has to be detected through other channels.

The following three statements, for example:

1. If the reader failed B, then he did not fail A,
2. If the reader did not fail B, then he failed C but not D,
3. Either the reader failed both A and B or he failed both A and D,

are sententially inconsistent and would prove to be inconsistent if subjected to the truth-table test. The following two statements, on the other hand:

1. All men are mortal,
2. Some men are not mortal,

though logically inconsistent, are not sententially inconsistent and hence would not prove to be inconsistent if subjected to the truth-table test.

Two points deserve mention in this respect:

1. When a set of statements is inconsistent, it logically implies any statement whatsoever. This point becomes of interest when the statements in question, say $\varphi_1, \varphi_2, \ldots, \varphi_n$, are offered as premises for a conclusion ψ. The conditional:

$$\ulcorner(\varphi \,.\, \varphi_2 \,.\, \ldots \,.\, \varphi_n) \supset \psi\urcorner,$$

is then logically true; but so is any other conditional opening with

$$\ulcorner \varphi_1 \, . \, \varphi_2 \, . \, \ldots \, . \, \varphi_n \urcorner$$

as antecedent. For example, the three statements listed above, if offered as premises for the conclusion: 'The reader failed C', would logically imply that conclusion, but they would also imply the conclusion: 'I am Julius Caesar'.

2. When a set of statements is inconsistent, it implies any statement whatsoever and its denial. This point becomes of interest again when the statements in question, say $\varphi_1, \varphi_2, \ldots, \varphi_n$, are offered as premises for a conclusion ψ. The conditional:

$$\ulcorner(\varphi_1 \, . \, \varphi_2 \, . \, \ldots \, . \, \varphi_n) \supset \psi\urcorner,$$

is then logically true; but so is the conditional:

$$\ulcorner(\varphi_1 \, . \, \varphi_2 \, . \, \ldots \, . \, \varphi_n) \supset \sim\psi\urcorner.$$

For example, the three statements listed above, if offered as premises for the conclusion: 'The reader failed C', would logically imply that conclusion, but they would also imply the conclusion: 'The reader did not fail C'.

In view of these two facts, it is wise, when testing a set of premises $\varphi_1, \varphi_2, \ldots, \varphi_n$ and a conclusion ψ for sentential deducibility, to check whether the antecedent of

$$\ulcorner(\varphi_1 \, . \, \varphi_2 \, . \, \ldots \, . \, \varphi_n) \supset \psi\urcorner$$

is assigned only 'F's, since in that case the premises are inconsistent.

We shall close this section with a survey of the so-called *modes of sentential composition*.

Let ψ be a given sentential schema. The various compounds φ_i in which ψ may enter as sole component will have one of the following four sets of truth-values:

ψ	φ_1	φ_2	φ_3	φ_4
T	T	F	T	F
F	F	T	T	F

Each one of these sets of truth-values may be taken to define a *singulary mode of sentential composition*. The first mode is trivial; it yields ψ. The second mode yields $\ulcorner\sim\psi\urcorner$, the denial of ψ. The third mode yields all tautologies formed out of ψ, like $\ulcorner\psi \vee \sim\psi\urcorner$. The fourth mode yields all contradictions formed out of ψ, like $\ulcorner\psi \, . \sim\psi\urcorner$.

Let ψ_1 and ψ_2 be two given sentential schemata. The various com-

pounds φ_i in which ψ_1 and ψ_2 may enter as sole components will have one of the following sixteen sets of truth-values:

ψ_1	ψ_2	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8
T	T	T	F	T	F	T	F	T	F
F	T	F	T	T	F	T	F	F	T
T	F	F	T	F	T	T	F	T	F
F	F	F	T	F	T	F	T	T	F

ψ_1	ψ_2	φ_9	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	φ_{15}	φ_{16}
T	T	T	F	T	F	T	F	T	F
F	T	T	F	F	T	F	T	T	F
T	F	F	T	F	T	T	F	T	F
F	F	T	F	T	F	F	T	T	F

Each one of these sets of truth-values may be taken to define a *binary mode of sentential composition.* The first mode yields $\ulcorner\psi_1 \,.\, \psi_2\urcorner$, the conjunction of ψ_1 and ψ_2. The fifth mode yields $\ulcorner\psi_1 \vee \psi_2\urcorner$, the inclusive alternation of ψ_1 and ψ_2. The ninth mode yields $\ulcorner\psi_1 \supset \psi_2\urcorner$, the conditional formed out of ψ_1 as antecedent and ψ_2 as consequent. The eleventh mode yields $\ulcorner\psi_1 \equiv \psi_2\urcorner$, the biconditional formed out of ψ_1 as left-hand component and ψ_2 as right-hand component. The twelfth mode yields $\ulcorner\psi_1 \not\equiv \psi_2\urcorner$, the exclusive alternation of ψ_1 and ψ_2. Modes fifteen and sixteen are also familiar; they respectively yield all tautologies formed out of ψ_1 and ψ_2, like $\ulcorner(\psi_1 \,.\, \psi_2) \supset \psi_1\urcorner$, and all contradictions formed out of ψ_1 and ψ_2, like $\ulcorner{\sim}((\psi_1 \,.\, \psi_2) \supset \psi_1)\urcorner$.

Of the nine remaining modes only three deserve attention, namely 2, 6, and 7. Mode two yields what we call *the alternative denial of* ψ_1 *and* ψ_2:

$$\ulcorner\text{not } \psi_1 \text{ or not } \psi_2\urcorner,$$

commonly abbreviated $\ulcorner\psi_1 \mid \psi_2\urcorner$. Mode six yields what we call *the joint denial of* ψ_1 *and* ψ_2:

$$\ulcorner\text{neither } \psi_1 \text{ nor } \psi_2\urcorner,$$

commonly abbreviated $\ulcorner\psi_1 \downarrow \psi_2\urcorner$. Mode seven yields what we call *the converse conditional of* ψ_1 *and* ψ_2:

$$\ulcorner\psi_1, \text{ if } \psi_2\urcorner,$$

sometimes abbreviated $\ulcorner\psi_1 \subset \psi_2\urcorner$.

There are thus 2^{2^1} ($= 4$) singulary and 2^{2^2} ($= 16$) binary modes of sentential composition. If we pursued our investigation, we would encounter 2^{2^3} ($= 256$) ternary, 2^{2^4} ($= 65{,}536$) quaternary, and, more

generally, 2^{2^n} n-ary modes of sentential composition. m-ary ($m > 2$) modes may be left aside, however, as they are all analyzable into singulary and binary modes.

*11. MANY-VALUED SENTENTIAL LOGICS

The sentential logic we studied above reckoned only two truth-values: 'T' and 'F'; it is accordingly called *two-valued sentential logic*. An infinite number of sentential logics have also been set up reckoning each a finite number n ($n > 2$) of truth-values, and sentential logics have been set up reckoning an infinite number of truth-values.[7] These sentential logics may be grouped under the common heading: 'many-valued logics'. Their truth-values are usually taken to be numerals rather than letters; this device may be extended to two-valued logic as well, where 'T' may give way to '1' and 'F' to '2'.

Many-valued logics raise a delicate problem of interpretation. In two-valued logic, we may remember, '1' and '2' were paired with the two metalogical predicates 'is true' and 'is false'. In many-valued logics, '1', '2', '3', and so on, may similarly be paired with metalogical predicates. The choice of these predicates is, however, a debated issue; we shall discuss it first in connection with three-valued logic.

Some logicians have proposed to pair '1', '2', and '3' with semantical predicates: '1' with 'is true', '2' with 'is neither true nor false', and '3' with 'is false'. Other logicians have proposed to pair them with pragmatical predicates: '1' with 'is known to be true', '2' with 'is neither known to be true nor known to be false', and '3' with 'is known to be false'. Among the logicians favoring the second proposal, some claim that a statement may have two semantical and three pragmatical truth-values at a time and hence that two-valued and three-valued logics are not alternative, but complementary, logics.

The same issue arises, of course, with respect to each n-valued logic ($n > 2$) and with respect to infinitely many-valued logics; it gets more complex, however, as the number of truth-values grows larger. In four-valued logic, for example, the following two proposals have been made:

1. to pair '1' with the semantical predicate 'is true';
   '2' " " " " 'is truer than false';
   '3' " " " " 'is falser than true';
   '4' " " " " 'is false';

[7]Finitely many-valued logics were independently developed by the Polish logician J. Łukasiewicz and the American logician E. L. Post in the early nineteen twenties; infinitely many-valued logics are due to Łukasiewicz and his Polish collaborator A. Tarski.

2. to pair '1' with the pragmatical predicate 'is known to be true';
'2' " " " " 'is more known to be true than known to be false';
'3' " " " " 'is more known to be false than known to be true';
'4' " " " " 'is known to be false'.

Let us remark, by the way, that four-valued logic has no analogue of the three-valued predicate 'is neither true nor false' or 'is neither known to be true nor known to be false'; five-valued logic, on the other hand, has such an analogue, namely: the predicate paired with '3'.

We shall not argue here which interpretation, the semantical or the pragmatical, should be favored in any case. We shall not argue, either, which sentential logic should be favored over its rivals. Many objections have been raised to two-valued logic; if, however, such objections have any weight, objections of equal weight may be raised to any logic reckoning a finite number of truth-values. A case can be made, on the other hand, for infinitely many-valued logics. Such logics replace the two classical truth-values by an infinite set of truth-values and hence view truth and falsehood somewhat as limits towards which converge all other truth-values.

The number of m-ary modes of sentential composition within each n-valued logic is n^{n^m}. There are thus 27 singulary and 19,683 binary modes of sentential composition in three-valued logic. The 27 singulary modes in question may be tabulated as follows:

Component				Compounds					
ψ	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
1	1	2	3	1	1	1	2	2	2
2	1	2	3	1	2	2	1	1	2
3	1	2	3	2	1	2	1	2	1

Component				Compounds					
ψ	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	φ_{15}	φ_{16}	φ_{17}	φ_{18}
1	1	1	1	3	3	3	2	2	2
2	1	3	3	1	1	3	2	3	3
3	3	1	3	1	3	1	3	2	3

Component	Compounds								
ψ	φ_{19}	φ_{20}	φ_{21}	φ_{22}	φ_{23}	φ_{24}	φ_{25}	φ_{26}	φ_{27}
1	3	3	3	1	1	2	2	3	3
2	2	2	3	2	3	1	3	1	2
3	2	3	2	3	2	3	1	2	1

From the 27 singulary modes of three-valued logic and, more generally, from the n^{n^1} singulary modes of n-valued logic, one may be selected to match the two-valued mode: denial. Similarly, from the 19,683 binary modes of three-valued logic and, more generally, from the n^{n^2} binary modes of n-valued logic, five may be selected to match the two-valued modes: conjunction, inclusive alternation, conditional, biconditional, and exclusive alternation. The ones most currently used have the following truth-values:

Let n be the number of truth-values of a given logic, 'i' the truth-value of a given component φ and 'j' the truth-value of a given component ψ; then:

1. $\ulcorner {\sim_n}\varphi \urcorner$ has the truth-value 'k', where $k = (n - i) + 1$;
2. $\ulcorner \varphi \cdot_n \psi \urcorner$ has the truth-value 'k', where k is the larger of i and j;
3. $\ulcorner \varphi \vee_n \psi \urcorner$ has the truth-value 'k', where k is the smaller of i and j;
4. $\ulcorner \varphi \supset_n \psi \urcorner$ has the truth-value 'k', where $k = 1$ for $i \geq j$ and $k = 1 + (j - i)$ for $i < j$;
5. $\ulcorner \varphi \equiv_n \psi \urcorner$ has the truth-value 'k', where $k = 1 + |\, i - j \,|$;[8]
6. $\ulcorner \varphi \not\equiv_n \psi \urcorner$ has the truth-value 'k', where $k = n - |\, i - j \,|$.[8]

If we work out these six formulae with $n = 2$, we get the familiar tables for '$\sim$', '$.$', '$\vee$', '$\supset$', '$\equiv$', and '$\not\equiv$'. If we work them out with $n = 3$, we get the following tables for '$\sim_3$', '$._3$', '$\vee_3$', '$\supset_3$', '$\equiv_3$', and '$\not\equiv_3$':
Three-valued denial:

Component φ	Compound $\ulcorner {\sim_3}\varphi \urcorner$
1	3
2	2
3	1

(This truth-table is thus the table for φ_{27} above);
Three-valued conjunction, inclusive alternation, conditional, biconditional, and exclusive alternation:

[8] $|\, i - j \,|$ is the so-called *absolute difference of i and j*, that is, the difference from the larger to the smaller of the two numbers i and j.

Components		Compounds				
φ	ψ	$\ulcorner\varphi\ ._3\ \psi\urcorner$	$\ulcorner\varphi \vee_3 \psi\urcorner$	$\ulcorner\varphi \supset_3 \psi\urcorner$	$\ulcorner\varphi \equiv_3 \psi\urcorner$	$\ulcorner\varphi \not\equiv_3 \psi\urcorner$
1	1	1	1	1	1	3
2	1	2	1	1	2	2
3	1	3	1	1	3	1
1	2	2	1	2	2	2
2	2	2	2	1	1	3
3	2	3	2	1	2	2
1	3	3	1	3	3	1
2	3	3	2	2	2	2
3	3	3	3	1	1	3

The techniques of section 10 may be used to compute the truth-values of three-valued and, more generally, n-valued ($n > 2$) formulae. We compute, for illustration's sake, the truth values of the three-valued schemata '$p \vee_3 \sim_3 p$' and '$(p\ ._3\ q) \equiv_3 (q\ ._3\ p)$':

p	$p \vee_3$	$\sim_3 p$
1	1	3
2	2	2
3	1	1

p	q	$(p\ ._3\ q)$	$\equiv_3$	$(q\ ._3\ p)$
1	1	1	1	1
2	1	2	1	2
3	1	3	1	3
1	2	2	1	2
2	2	2	1	2
3	2	3	1	3
1	3	3	1	3
2	3	3	1	3
3	3	3	1	3

An *n-valued tautology* may be defined, in analogy with a two-valued tautology, as an n-valued schema whose truth-value is '1' whatever be the truth-values of its components.[9] A detailed parallel between two-valued and n-valued tautologies cannot be undertaken here; two points, however, are worth mentioning.

1. Some two-valued tautologies have analogues in all n-valued logics; such is the case with

$$(p\ .\ q) \equiv (q\ .\ p),$$

[9]The truth-values which a formula may assume if it is to qualify as a tautology are technically called *designated values*. Some logicians have proposed to enlarge the number of designated values in n-valued ($n > 2$) logics from 1 to any arbitrary number m such that $m < n$. Their proposal dims the epistemological significance of n-valued tautologies; it is accordingly not adopted here.

which may be generalized into:

$$(p \cdot_n q) \equiv_n (q \cdot_n p).$$

All sentential logics thus have a common stock of tautologies; this stock could be isolated.

2. Some two-valued tautologies, on the other hand, have no analogues in any n-valued logic; such is the case with

$$p \vee \sim p.$$

Two-valued logic thus has a proper stock of tautologies, which could also be isolated.

*12. MODAL SENTENTIAL LOGICS

A sentential compound is said to be *truth-functional* when its truth-values are determined by the truth-values of its components, and *non-truth-functional* when its truth-values are not determined by the truth-values of its components. The above logics admitted only truth-functional compounds. Modal sentential logics, on the other hand, admit non-truth-functional compounds like:

It is necessary that p (1),
It is not necessary that p (2),[10]
It is possible that p (3),

and

It is not possible that p (4).[11]

The analogy between these and the compounds:

It is the case that p ·

and

It is not the case that p,

of two-valued sentential logic, is brought out clearly by the following variants of (1)–(4):

It is necessarily the case that p,
It is not necessarily the case that p,
It is possibly the case that p,

and

It is not possibly the case that p.

[10]Some logicians have used 'contingent' in place of 'not necessary'; note, however, that 'It is contingent that p' has also been understood as 'It is possible but not necessary that p'.

[11]We write 'not possible' rather than 'impossible' to avoid using below the awkward phrase 'It is impossibly the case that p'.

The American logician C. I. Lewis has introduced the sign '$\Diamond$' as an abbreviation for 'It is possible that'.[12] If we understand 'It is necessary that p' as 'It is not possible that not p' (and hence 'It is not necessary that p' as 'It is possible that not p'), we may accordingly abbreviate (1)–(4) into:

$$(1): \sim\Diamond\sim p,$$

$$(2): \Diamond\sim p,$$

$$(3): \Diamond p,$$

and

$$(4): \sim\Diamond p.$$

Modal compounds are especially significant because of the metalogical translations of which they are susceptible. Let us approach this problem from the standpoint of two-valued sentential logic. The truth-functional compounds φ and $\ulcorner\sim\varphi\urcorner$, where φ is a statement, may respectively be translated into the following two metalogical statements:

φ is true

and

φ is false.

For instance, the statement 'Snow is white' may be translated into the metalogical statement: ' 'Snow is white' is true'; and the statement '$\sim$(Coal is white)' may be translated into the metalogical statement: ' 'Coal is white' is false'.

Similarly, the modal compounds $\ulcorner\sim\Diamond\sim\varphi\urcorner$, $\ulcorner\sim\Diamond\varphi\urcorner$, $\ulcorner\Diamond\varphi\urcorner$, and $\ulcorner\Diamond\sim\varphi\urcorner$, where φ is a statement, may respectively be translated into the following four metalogical statements:

φ is logically true,
φ is logically false,
φ is not logically false,

and

φ is not logically true.

For instance, '$\sim\Diamond\sim$(John is sick $\supset$ John is sick)' may be translated into: ' 'John is sick $\supset$ John is sick' is logically true'; '$\sim\Diamond$(John is sick . $\sim$(John is sick))' may be translated into : ' 'John is sick . $\sim$(John is sick)' is logically false'; '$\Diamond$(John is sick)' may be translated into: ' 'John is sick' is not logically false'; and '$\Diamond\sim$(John is sick)' may be translated into: ' 'John is sick' is not logically true'.

[12]C. I. Lewis is to be credited with the revival of modal sentential logics in the early nineteen tens: modal compounds were well-known to Aristotle and the Medievals.

We thus obtain the following table of translations:

Table one

	Metalogical translations
φ:	φ is true;
$\ulcorner \sim\varphi \urcorner$:	φ is false;
$\ulcorner \sim\Diamond\sim\varphi \urcorner$:	φ is logically true;
$\ulcorner \sim\Diamond\varphi \urcorner$:	φ is logically false;
$\ulcorner \Diamond\varphi \urcorner$:	φ is not logically false;
$\ulcorner \Diamond\sim\varphi \urcorner$:	φ is not logically true.

Several additions can be made to this table.

(*a*) If we take a statement to be factually true when it is true but is not logically true, and a statement to be factually false when it is false but is not logically false, then we obtain the following two extra translations:

Table two

	Metalogical translations
$\ulcorner \varphi \,.\, \Diamond\sim\varphi \urcorner$:	φ is factually true;
$\ulcorner \sim\varphi \,.\, \Diamond\varphi \urcorner$:	φ is factually false.

(*b*) If, as we did above, we take a statement φ to logically imply a statement ψ when the conditional $\ulcorner \varphi \supset \psi \urcorner$ is logically true, and take a statement φ to be logically equivalent to a statement ψ when the biconditional $\ulcorner \varphi \equiv \psi \urcorner$ is logically true, then we obtain the following two extra translations:

Table three

	Metalogical translations
$\ulcorner \sim\Diamond\sim(\varphi \supset \psi) \urcorner$:	φ logically implies ψ;
$\ulcorner \sim\Diamond\sim(\varphi \equiv \psi) \urcorner$:	φ is logically equivalent to ψ.

Special abbreviations have been provided for $\ulcorner \sim\Diamond\sim(\varphi \supset \psi) \urcorner$ and $\ulcorner \sim\Diamond\sim(\varphi \equiv \psi) \urcorner$, namely: $\ulcorner \varphi \prec \psi \urcorner$ and $\ulcorner \varphi \equiv \psi \urcorner$; '$\prec$' is sometimes called *the strict* or *logical implication sign,* '$\equiv$' *the strict* or *logical equivalence sign.*

(*c*) Finally, if, as we did above, we take a set of statements to be logically consistent when their conjunction is not logically false and to be logically inconsistent when their conjunction is logically false, then we obtain the following two extra translations:

Table four

	Metalogical translations
$\ulcorner \Diamond(\varphi_1 \,.\, \varphi_2 \,.\, \ldots \,.\, \varphi_n) \urcorner$:	$\varphi_1, \varphi_2, \ldots, \varphi_n$ are logically consistent;
$\ulcorner \sim\Diamond(\varphi_1 \,.\, \varphi_2 \,.\, \ldots \,.\, \varphi_n) \urcorner$:	$\varphi_1, \varphi_2, \ldots, \varphi_n$ are logically inconsistent.

A special abbreviation has been provided for $\ulcorner\Diamond(\varphi_1 \,.\, \varphi_2 \,.\, \ldots \,.\, \varphi_n)\urcorner$, namely: $\ulcorner\varphi \circ \varphi_2 \circ \ldots \circ \varphi_n\urcorner$; '$\circ$' is sometimes called *the logical consistency sign.*

Table one shows clearly why the four modal compounds $\ulcorner\sim\Diamond\sim\varphi\urcorner$, $\ulcorner\sim\Diamond\varphi\urcorner$, $\ulcorner\Diamond\varphi\urcorner$, and $\ulcorner\Diamond\sim\varphi\urcorner$ are not truth-functional. It follows from the definition of a truth-functional compound that if two statements φ and ψ have the same truth-value, then the two truth-functional compounds:

$$\ulcorner\text{———}\varphi\text{———}\urcorner$$

and

$$\ulcorner\text{———}\psi\text{———}\urcorner,$$

also have the same truth-value. For instance, since the two statements: 'Trenton is south of New York' and 'Trenton is south of New York $\supset$ Trenton is south of New York', have the same truth-value, the two statements: '$\sim$(Trenton is south of New York)' and '$\sim$(Trenton is south of New York $\supset$ Trenton is south of New York)', also have the same truth-value.

But two statements φ and ψ may very well have the same truth-value without the two metalogical statements:

φ is logically true

and

ψ is logically true,

having the same truth-value. For instance, the two statements: 'Trenton is south of New York' and 'Trenton is south of New York $\supset$ Trenton is south of New York', have, as noted above, the same truth-value; yet the metalogical statement:

'Trenton is south of New York' is logically true,

is false, while the metalogical statement:

'Trenton is south of New York $\supset$ Trenton is south of New York' is logically true,

is true.

If, in accordance with table one, we take 'φ is logically true' to be a translation of $\ulcorner\sim\Diamond\sim\varphi\urcorner$, we are accordingly left with two statements: 'Trenton is south of New York' and 'Trenton is south of New York $\supset$ Trenton is south of New York', which have the same truth-value, whereas the two modal compounds:

$\sim\Diamond\sim$(Trenton is south of New York)

and

$\sim\Diamond\sim$(Trenton is south of New York $\supset$ Trenton is south of New York), have different truth-values.

Modal sentential logics may be defined as the logics which result from adding the possibility sign '$\Diamond$' to the twelve signs 'p', 'q', 'r', 's', '$\sim$', '.', 'v', '$\not\equiv$', '$\subset$', '$\equiv$', '(', and ')' of two-valued sentential logic. There exist a number of modal sentential logics, differing among themselves in the schemata they include or the schemata they acknowledge as valid. We shall study one of them here under the label 'M'; its schemata, called *M-schemata*, may be described as follows:

1. A two-valued sentential schema is an M-schema;
2. The result of writing '$\Diamond$' in front of a two-valued sentential schema is an M-schema;
3. The negation of an M-schema is an M-schema;
4. The conjunction, alternation, conditional, or biconditional of two M-schemata is an M-schema.[13]

M-schemata fall into three groups:

1. valid M-schemata like '$\sim\Diamond\sim p \supset p$', all instances of which are true;
2. indeterminate M-schemata like '$p \supset \sim\Diamond\sim p$', some instances of which are true and some instances of which are false;
3. contravalid M-schemata like '$\sim(\sim\Diamond\sim p \supset p)$', all instances of which are false.

There exists a mechanical procedure for identifying valid M-schemata; it is not, however, of the tabulation type. We shall content ourselves here with listing twenty valid M-schemata.

M1: $\sim\Diamond\sim p \supset p$;
M2: $p \supset \Diamond p$;
M3: $\sim\Diamond p \supset \sim p$;
M4: $\sim p \supset \Diamond\sim p$.

It follows from M1–M4 that if a statement is logically true, then it is true; that if a statement is true, then it is not logically false; that if a statement is logically false, then it is false; and that if a statement is false, then it is not logically true.

M5: $p \supset (\sim\Diamond\sim p \vee (p \,.\, \Diamond\sim p))$;
M6: $\sim p \supset (\sim\Diamond p \vee (\sim p \,.\, \Diamond p))$.

[13] 1–4 constitute what we shall call in chapter three a *recursive definition* of the phrase 'M-schema'.

It follows from M5–M6 that if a statement is true, then it is either logically or factually true; and that if a statement is false, then it is either logically or factually false.

M7: $\sim\Diamond\sim p \equiv (\sim p \prec p)$;
M8: $\sim\Diamond p \equiv (p \prec \sim p)$;
M9: $\Diamond p \equiv \sim(p \prec \sim p)$;
M10: $\Diamond\sim p \equiv \sim(\sim p \prec p)$.

It follows from M7–M10 that a statement is logically true if and only if it is logically implied by its own denial; that a statement is logically false if and only if it logically implies its own denial; and so on.

M11: $\sim\Diamond\sim(p \,.\, q) \equiv (\sim\Diamond\sim p \,.\, \sim\Diamond\sim q)$;
M12: $\Diamond(p \vee q) \equiv (\Diamond p \vee \Diamond q)$;
M13: $\Diamond(p \,.\, q) \supset (\Diamond p \,.\, \Diamond q)$;
M14: $(\sim\Diamond\sim p \vee \sim\Diamond\sim q) \supset \sim\Diamond\sim(p \vee q)$.

It follows from M11–M14 that a conjunction is logically true if and only if each one of its factors is logically true; that an alternation is not logically false if and only if at least one of its summands is not logically false; and so on. It is easily seen that the converses of M13–M14 are not valid. Neither 'Peter is ten years old' nor 'Peter is eleven years old' is logically false; yet their conjunction: 'Peter is ten years old . Peter is eleven years old', is logically false. Similarly, the alternation: 'Peter is ten years old v Peter is not ten years old', is logically true; yet neither 'Peter is ten years old' nor 'Peter is not ten years old' is logically true.

M15: $((p \prec q) \,.\, \sim\Diamond\sim p) \supset \sim\Diamond\sim q$;
M16: $((p \prec q) \,.\, \Diamond p) \supset \Diamond q$;
M17: $((p \prec q) \,.\, \sim\Diamond q) \supset \sim\Diamond p$;
M18: $((p \prec q) \,.\, \Diamond\sim q) \supset \Diamond\sim p$.

It follows from M15-M18 that if a given statement is logically true, then whatever statement it logically implies is also logically true; that if a given statement is not logically false, then whatever statement it logically implies is not logically false either; and so on.

M19: $(\sim\Diamond\sim p \,.\, \sim\Diamond\sim q) \supset (p \equiv q)$;
M20: $(\sim\Diamond p \,.\, \sim\Diamond q) \supset (p \equiv q)$.

It follows from M19–M20 that if two statements are both logically true or both logically false, then they are logically equivalent.

CHAPTER

T W O

Quantificational Logic

13. THE LOGISTIC ANALYSIS OF STATEMENTS

We have seen in chapter one how statements combine into further statements; we shall now see how nouns, predicates, and indefinite adjectives combine into statements. Since the two phrases 'all' and 'some', technically called *quantifiers*, play a crucial role in that operation, the area of logic we are to survey is often termed *quantificational logic*. Many-valued and modal quantificational logics have recently been set up; they are less well-known, however, than their sentential counterparts and will not be studied here. A section is appended to the main body of the chapter on so-called *intuitionist logic*.

All singular statements include a subject and a verb. These two units of discourse are independent; one verb may, for instance, be preceded by different subjects, like 'to come' in

John comes

and

Peter comes;

and one subject may precede different verbs, like 'John' in

John comes

and

John leaves.

This rough dichotomy: subject and verb, is exhaustive when the verb, as in the above examples, is intransitive. When the verb is transitive, however, another unit of discourse appears: the object. There are, we know, two types of objects: direct objects and indirect ones. Some transitive verbs take only direct objects, like 'to love' in

June loves Van Johnson;

others take only indirect objects, like 'to go' in

June often goes to the movies;

others, lastly, take both direct and indirect objects, like 'to give' in

I give money to the Community Chest.

A transitive verb may, of course, take any number of objects, direct and/or indirect, as 'to give' does in

I give clothes and money to the Community Chest and the Red Cross.

The various ingredients which make up a singular statement are grouped by logic into two classes:

1. subjects and objects, also called *arguments*;
2. verbs, also called *predicates*.

This classification seems to leave out many particles which also occur in singular statements, like adverbs, prepositions, and conjunctions. Adverbs and prepositions may be handled as fragments of predicates. The adverb 'often' and the preposition 'to' in

I often give money to the Community Chest

may, for example, be treated as fragments of the predicate 'often-give-to'. The notion of a predicate is thus a broad one; it covers whatever is left of a statement when its subjects and objects are deleted as arguments.[1]

Conjunctions call for a more subtle treatment.

1. When the singular statement in which they occur can be paraphrased as a molecular one, then they may be handled as statement connectives. For example,

Paul and John came

and

Paul went to Boston and New York

can be paraphrased as:

Paul came and John came

and

Paul went to Boston and Paul went to New York.

In both cases 'and' may therefore be rendered as '.'.

2. When the singular statement in which they occur cannot be paraphrased as a molecular one, then they must be handled as fragments of predicates. For example,

Paul and John came together (1)

[1]We shall later broaden this notion even more and understand by a *predicate* whatever is left of a statement when at least one of its subjects or objects is deleted as argument.

and

Paul visited Boston and New York the same day (2)

can obviously not be paraphrased as:

Paul came together and John came together

and

Paul visited Boston the same day and Paul visited New York the same day.

(1) and (2) must therefore be viewed as atomic statements arising from the insertion of the arguments 'Paul', 'John', 'Boston', and 'New York' in the blanks of

—— and —— came together

and

—— visited —— and —— the same day.[2]

The foregoing analysis of statements into arguments and predicates is proper to modern logic. Aristotelian logic favored a different analysis whereby statements were dissected into a subject, a *copula* (the verb 'is'), and an attribute. Its analysis suited well some statements like:

Snow is white

and

John is coming,

which show a *copula* and a single argument. It clashed with others like:

John runs (3)

and

Harry sits between Paula and Betty (4),

which show no *copula* or show many arguments. To save its trichotomy, Aristotelian logic had to insert *ad hoc copulae* into the latter statements, thus paraphrasing (3) and (4) as:

John is running

and

Harry is sitting between Paula and Betty.

The Aristotelian analysis has, among others, two defects: it gives an undue prominence to the verb 'is' without, by the way, discriminating between its various uses; it breaks the line between one-place and many-place predicates. In accord with our own analysis, we shall treat

[2]When conjunctions enter a general statement, as 'and' does in

All men and women have registered,

they obey formal laws of their own to be listed in section 17.

the copulative phrases 'is white', 'is coming', and so on, of Aristotelian logic as complex predicates whose fragments may, for clarity's sake, be hyphenated.[3]

As we needed above place-holders for statements, so we shall need here place-holders for predicates and arguments. We adopt to that effect two new sets of Latin letters: the argument letters 'w', 'x', 'y', and 'z', and the predicate letters 'F', 'G', and 'H'.[4] As the process whereby a sentential letter is replaced by one of its instances was called *sentential substitution,* so the process whereby an argument letter is replaced by one of its instances will be called *argument substitution* and the process whereby a predicate letter is replaced by one of its instances will be called *predicate substitution.*

Predicate and argument letters may combine to yield what we call *atomic quantificational schemata* like '$F(x)$', '$F(x,y)$', '$G(y,x,z)$', and so on. These schemata will admit as their instances all the statements which may be obtained by substituting predicates for their predicate letters and arguments for their argument letters. 'John reads', for example, may be obtained by substituting 'reads' for 'F' and 'John' for 'x' in '$F(x)$'; it accordingly qualifies as an instance of '$F(x)$'. 'Harry dislikes Tim', on the other hand, may be obtained by substituting 'dislikes' for 'F', 'Harry' for 'x', and 'Tim' for 'y' in '$F(x,y)$'; it accordingly qualifies as an instance of '$F(x,y)$'. 'John sits between Paul and James', finally, may be obtained by substituting 'sits between . . . and' for 'G', 'John' for 'y', 'Paul' for 'x', and 'James' for 'z' in '$G(y,x,z)$'; it accordingly qualifies as an instance of '$G(y,x,z)$'.

The following two points should be borne in mind:

1. An atomic quantificational schema may contain more argument places than some of its instances do;
2. An atomic quantificational schema may contain less argument places than some of its instances do.

The first point, of little practical significance, will be taken up in section 18. As for the second, note that the statement 'Harry dislikes Tim' may be obtained from '$F(x)$' in either one of two ways: by substituting 'dislikes Tim' for 'F' and 'Harry' for 'x' in '$F(x)$' or by substituting 'Harry dislikes' for 'F' and 'Tim' for 'x' in '$F(x)$'; it accordingly is an instance of '$F(x)$' as well as of '$F(x,y)$'.

[3]The word 'logistic', under which modern logic has long been known, was used in the title of section 13 to accentuate the contrast between the modern analysis of statements and the Aristotelian one.

[4]Further argument and predicate letters may be manufactured by adding accents to the seven letters 'w', 'x', 'y', 'z', 'F', 'G', and 'H'. The same remark applies retroactively to 'p', 'q', 'r', and 's'.

Atomic quantificational schemata may, like sentential letters, be governed by connectives; the resulting compounds, '$\sim F(x)$', '$F(x) \supset G(x)$', '$F(x,y) \equiv H(w,z,y)$', for example, are called *molecular quantificational schemata.* Their instances are molecular statements. '$\sim$(John is sick)', 'Socrates is a man $\supset$ Socrates is mortal', and 'John will invite Paul $\equiv$ James agrees to sit between Harry and Paul', for example, are instances of '$\sim F(x)$', '$F(x) \supset G(x)$', and '$F(x,y) \equiv H(w,z,y)$', respectively. Let us remark that whereas molecular quantificational schemata admit only molecular statements as their instances, atomic quantificational schemata admit both atomic and molecular statements as theirs. The molecular schema '$F(x) \supset G(x)$', for example, admits as its instances only molecular statements like 'Socrates is a man $\supset$ Socrates is mortal'. The atomic schema '$F(x)$', on the other hand, admits as its instances both atomic statements like 'Socrates was a Greek' and molecular ones like 'Socrates is a man $\supset$ Socrates is mortal'. The first instance may be obtained by substituting 'was a Greek' for 'F' and 'Socrates' for 'x' in '$F(x)$', the second by substituting 'is a man $\supset$ . . . is mortal' for 'F' and 'Socrates' for 'x' in '$F(x)$'.

A large number of statements arise by substitution from quantificational schemata; they are all, however, *singular statements.* We still have to account for the formation of *general statements*: *universal statements* like 'All men are mortal' and *particular statements* like 'Some musicians are colored'.

14. QUANTIFIERS

Let us consider the quantificational schema: '$F(x) \supset G(x)$'. If we substitute the predicate 'is a man' for 'F' and the predicate 'is mortal' for 'G', '$F(x) \supset G(x)$' yields:

$$x \text{ is a man} \supset x \text{ is mortal} \qquad (1).$$

This conditional is not yet a statement. If, however, we go on substituting for 'x' various arguments like 'Socrates', 'Plato', 'Harry James', and so on, then (1) yields one true statement after another, namely:

Socrates is a man $\supset$ Socrates is mortal,
Plato is a man $\supset$ Plato is mortal,
Harry James is a man $\supset$ Harry James is mortal,

and so on. To the question: "How many x's satisfy (1)?" the answer is thus: "All." We may accordingly write 'For all x' in front of (1); the resulting expression:

For all x, x is a man $\supset$ x is mortal,

is a statement, condensed in everyday language into:

All men are mortal.

The phrase 'For all x' will be our first statement-forming operator; it turns quantificational schemata into universal statements.

Let us next consider the quantificational schema: '$F(x) \,.\, G(x)$'. If we substitute the predicate 'is a musician' for 'F' and the predicate 'is colored' for 'G', '$F(x) \,.\, G(x)$' yields:

x is a musician . x is colored (2).

This conjunction is not yet a statement. If, however, we go on substituting for 'x' various arguments like 'Harry James', 'Artie Shaw', 'Louis Armstrong', 'Duke Ellington', and so on, then (2) yields some false statements like:

Harry James is a musician . Harry James is colored,
Artie Shaw is a musician . Artie Shaw is colored,

and so on, but it also yields some true statements like:

Louis Armstrong is a musician . Louis Armstrong is colored,
Duke Ellington is a musician . Duke Ellington is colored,

and so on. To the question: "How many x's satisfy (2)?" the answer is thus: "Some." We may according write 'For some x' in front of (2); the resulting expression:

For some x, x is a musician . x is colored,

is a statement, condensed in everyday language into:

Some musicians are colored.

The phrase 'For some x' will be our second statement-forming operator; it turns quantificational schemata into particular statements.

We may thus conclude that there are two main ways of obtaining statements from a quantificational schema:

1. By substituting predicates and arguments for its predicate and argument letters; this process yields singular statements.
2. By substituting predicates for its predicate letters and prefixing to the schema one of the two phrases: 'For all x' and 'For some x'; this process yields general statements, that is, universal and particular statements. The two phrases: 'For all x' and 'For some x', are usually called *quantifiers*, and the process of writing them in front of a schema *quantification*. Our two modes of statement formation may thus be identified as substitution and quantification.

We shall respectively abbreviate the two phrases: 'For all x' and 'For some x', as '(x)' and '$(\mathrm{E}x)$. '(x)' is called a *universal*, '$(\mathrm{E}x)$' a *particular quantifier*; since '$(\mathrm{E}x)$', however, is often read: 'There exists some x such that', it is also called an *existential quantifier*. Let us note, in connection with these two readings of '$(\mathrm{E}x)$', that the word 'some' is used very ambiguously in daily conversation. When used in the singular, it often stands for 'exactly one'; for example, 'Some man came' must be interpreted as 'Exactly one man came'. When used in the plural, it usually stands for 'a few', that is, 'more than one'; for example, 'Some books are left on the shelf' must be interpreted as 'A few books are left on the shelf'. In logic, 'some' has a set meaning: *at least one*. The quantifier '$(\mathrm{E}x)$' may thus be read: 'There exists at least one x such that'. In chapter 4 we shall devise numerical quantifiers reading: 'There exists at most one x', 'There exists exactly one x', 'There exist at least two x', 'There exist at most two x', 'There exist exactly two x', and so on. These quantifiers will account for all the meanings of 'some'; for the time being, though, we may be satisfied with one existential quantifier, the least committal of them all: '$(\mathrm{E}x)$'.

Our study of universal and existential quantifiers may now be extended in four significant ways:

1. We have quantified so far only the argument letter 'x'; our remaining three argument letters may be quantified as well, though. We thus get the following set of quantifiers: '(w)', '(x)', '(y)', '(z)', '$(\mathrm{E}w)$', '$(\mathrm{E}x)$', '$(\mathrm{E}y)$', and '$(\mathrm{E}z)$'.

2. We have attached quantifiers so far to two main schemata: '$F(x) \supset G(x)$' and '$F(x) \,.\, G(x)$'; we can attach them, however, to any quantificational schema whatsoever, whether atomic or molecular. As sample quantified schemata let us quote:

'$(x)F(x)$', which may be instantiated as '(x)(x is self-identical)', that is, 'Everything is self-identical';
'$(x){\sim}F(x)$', which may be instantiated as '$(x){\sim}$(x is self-identical)', that is, 'Nothing is self-identical';
'$(\mathrm{E}y)F(y)$', which may be instantiated as '$(\mathrm{E}y)$(y is cooking)', that is, 'Something is cooking';
'$(\mathrm{E}y){\sim}F(y)$', which may be instantiated as '$(\mathrm{E}y){\sim}$(y is cooked yet)', that is, 'Something is not cooked yet';
'$(z)(F(z) \,.\, G(z))$', which may be instantiated as '(z)(z annoys me . z infuriates me)', that is, 'Everything annoys and infuriates me';
'$(\mathrm{E}x)(F(x) \vee G(x))$', which may be instantiated as '$(\mathrm{E}x)$(x is broken $\vee$ x is jammed)', that is 'Something is broken or jammed'.

'$(x)(F(x) \supset G(x))$' and '$(\mathrm{E}x)(F(x) \,.\, G(x))$' are thus simply two among an

infinite number of quantified schemata; the general statements they yield, however, are key statements in everyday language.

3. Let us also remark that quantified as well as unquantified schemata may be governed by connectives, as shown by the following examples: '$\sim(x)F(x)$', '$\sim(\mathrm{E}x)F(x)$', '$(y)F(y) \supset (y)G(y)$', '$(z)F(z) \,.\, (\mathrm{E}y)G(y)$', and so on.

'$\sim(x)F(x)$' may be instantiated as '$\sim(x)(x$ is worth reading)', that is, 'Not everything is worth reading';
'$\sim(\mathrm{E}x)F(x)$' may be instantiated as '$\sim(\mathrm{E}x)(x$ excites you)', that is, 'Nothing excites you';
'$(y)F(y) \supset (y)G(y)$' may be instantiated as '$(y)(y$ is stressed alike) $\supset$ $(y)(y$ loses its significance)', that is, 'If everything is stressed alike, then everything loses its significance';
'$(z)F(z) \,.\, (\mathrm{E}y)G(y)$' may finally be instantiated as '$(z)(z$ is on this shelf $\supset$ z is worth saving) . $(\mathrm{E}y)(y$ is on that shelf . y may be thrown away)', that is, 'Everything on this shelf is worth saving, but something on that shelf may be thrown away'.

4. Let us finally remark that any number of quantifiers may be written in front of a given schema. '$F(x,y)$', for example, may be quantified into:

$$(x)(y)F(x,y),$$
$$(x)(\mathrm{E}y)F(x,y),$$
$$(\mathrm{E}y)(x)F(x,y),$$
$$(\mathrm{E}x)(\mathrm{E}y)F(x,y),$$

and so on. If we substituted the predicate 'is similar to' for 'F', we could turn these four schemata into four statements, namely:

Everything is similar to everything;
Everything is similar to something or other;
There exists something to which everything is similar;
Something is similar to something or other.

Quantifiers raise a further problem: the problem of *bondage* and *freedom*; before turning to it, however, we may introduce the auxiliary notion of a scope. *The scope of a quantifier* is the schema which that quantifier governs; the scope of '(x)' in '$(x)\sim F(x)$', for instance, is '$\sim F(x)$'; the scope of '$(\mathrm{E}x)$' in '$(\mathrm{E}x)(F(x) \supset G(x)) \,.\, F(x)$' is '$F(x) \supset G(x)$'; and the scope of '$(x)$' in '$(y)(F(y) \supset (x)(H(x) \,.\, G(y)))$' is '$H(x) \,.\, G(y)$'. Scopes, when showing only one component, as '$\sim F(x)$' does in '$(x)\sim F(x)$', may be left free of parentheses; when showing more than one component, however, as '$F(x) \supset G(x)$' does in '$(\mathrm{E}x)(F(x) \supset G(x)) \,.\, F(x)$' and

'$H(x)\ .\ G(y)$' does in '$(y)(F(y) \supset (x)(H(x)\ .\ G(y)))$', they must be enclosed within parentheses.

We shall say that *a given occurrence of an argument letter*, say 'x', *is bound* when that occurrence falls within one of the two quantifiers '(x)' and '$(\mathrm{E}x)$' or within the scope of one of the two quantifiers '(x)' and '$(\mathrm{E}x)$'. For instance, the quantificational schema:

$$(x)(F(x) \supset G(x)) \qquad (3),$$

contains three occurrences of the argument letter 'x'; all three of these occurrences are bound, the first one because it falls within '(x)', the last two because they fall within the scope of '(x)'. The quantificational schema:

$$(\mathrm{E}x)F(x) \supset F(x) \qquad (4),$$

also contains three occurrences of the argument letter 'x'; the first two of these occurrences are bound, one because it falls within '$(\mathrm{E}x)$', the other because it falls within the scope of '$(\mathrm{E}x)$; the third one of these occurrences, however, is not bound, since it neither falls within a quantifier nor within the scope of a quantifier.

We shall next say that *a given argument letter is bound in a given quantificational schema* when at least one of its occurrences in that schema is bound; the letter 'x', for example, is bound both in (3) and in (4).

We shall finally say that *a given occurrence of an argument letter in a given quantificational schema is free* if it is not bound in that schema, and that *a given argument letter is free in a given quantificational schema* if at least one of its occurrences is free in that schema. The third occurrence of the argument letter 'x' in (4), for example, is free and, hence, the argument letter 'x' itself is free in (4).

It follows from the above definitions that a given argument letter may be both bound and free in a given quantificational schema; each one of its occurrences, however, is either bound or free, but not both.

A quantificational schema like:

$$(y)(x)(F(x) \supset F(y)) \qquad (5),$$

no argument letter of which is free, may be called a *closed schema*; a quantificational schema like:

$$(x)(F(x) \supset F(y)) \qquad (6),$$

some argument letter of which is free, may be called an *open schema*. An open schema will automatically be turned into a closed one if a series of universal quantifiers, binding each one of its free argument letters, is written in front of it. For instance, the open schema (6) is automatically

turned into a closed schema, namely (5), if the universal quantifier '(y)', binding its free argument letter 'y', is written in front of (6). Similarly, the open schema:

$$(x)(F(x) \supset G(y,z)) \qquad (7),$$

is automatically turned into a closed one if the two universal quantifiers '$(y)(z)$' or the two universal quantifiers '$(z)(y)$' are written in front of it:

$$(y)(z)(x)(F(x) \supset G(y,z)) \qquad (8),$$

$$(z)(y)(x)(F(x) \supset G(y,z)) \qquad (9).$$

Any result of universally quantifying all the free argument letters of a given quantificational schema will be called *a closure of that schema;* (5) is thus a closure of (6), while (8) and (9) are two closures of (7).

Half-way between quantificational schemata like '$(x)(F(x) \supset G(x))$' and '$(x)(\mathrm{E}y)F(x,y)$', on one hand, and full-fledged statements like '$(x)(x$ is a man $\supset$ x is mortal)' and '$(x)(\mathrm{E}y)(x = y)$', on the other, stand expressions like:

$$x \text{ is a man} \supset x \text{ is mortal}$$

and

$$(\mathrm{E}y)(x = y),$$

whose closures:

$$(x)(x \text{ is a man} \supset x \text{ is mortal})$$

and

$$(x)(\mathrm{E}y)(x = y),$$

are statements. We shall call such expressions *quasi-statements;* they will play an important part in so-called *quantificational deduction.*

From now on we shall group schemata (both sentential and quantificational schemata), statements, and quasi-statements under the common heading 'formulae', and use the four metalogical letters 'φ', 'χ', 'ψ', and 'ω' as place-holders for formulae in this enlarged sense of the word 'formulae'.

15. VARIABLES AND DUMMIES

A place-holder which is subject to quantification is often called a *variable;* a place-holder which is not is called by contrast a *dummy.* The four letters 'w', 'x', 'y', and 'z', for instance, are subject to quantification; they may accordingly be treated as variables. The seven letters 'p', 'q', 'r', 's', 'F', 'G', and 'H', on the other hand, are not subject to quantification; they may accordingly be treated as dummies.

The distinction between variables and dummies has its semantical significance: the instances of a variable are normally expected to be

names, whereas the instances of a dummy are not. To illustrate this point, let us first consider the four letters 'w', 'x', 'y', and 'z'. In section 17, below, we shall pronounce all instances of the formula:

$$F(y) \supset (\mathrm{E}x)F(x) \qquad (1),$$

logically true. Now take the following instance of '$F(y)$':

Socrates is mortal,

where the argument 'Socrates' has been substituted for 'y'. In virtue of (1) 'Socrates is mortal' logically implies:

There exists an entity x such that x is mortal.

But 'Socrates is mortal' cannot logically imply 'There exists an entity x such that x is mortal' unless the argument 'Socrates' in 'Socrates is mortal' serves a name of some entity or other.

Consider next the three letters 'F', 'G', and 'H'. If we quantified 'F', 'G', and 'H', and acknowledged as logically true all instances of the formula:

$$\text{——} \; G \; \text{——} \supset (\mathrm{E}F)(\text{——} \; F \; \text{——}) \qquad (2),$$

then we would have to treat predicates as names. Take indeed the following instance of '—— G ——':

Socrates is mortal,

where the predicate 'is mortal' has been substituted for 'G'. In virtue of (2) 'Socrates is mortal' logically implies:

There exists an entity G such that G(Socrates).

But 'Socrates is mortal' cannot logically imply 'There exists an entity G such that G(Socrates)' unless the predicate 'is mortal' in 'Socrates is mortal' serves as a name of some entity or other.[5]

Consider finally the four letters 'p', 'q', 'r', and 's'. If we quantified 'p', 'q', 'r', and 's', and acknowledged as logically true all instances of the formula:

$$\text{——} \; q \; \text{——} \supset (\mathrm{E}p)(\text{——} \; p \; \text{——}) \qquad (3),$$

then we would have to treat statements as names. Take indeed the following instance of '—— q ——':

Socrates is a man ⊃ Socrates is mortal,

[5]The entity in question is usually taken to be a property, here the property being mortal.

where the statement 'Socrates is mortal' has been substituted for 'q'. In virtue of (3) 'Socrates is a man $\supset$ Socrates is mortal' logically implies:

There exists an entity p such that Socrates is a man $\supset$ p.

But 'Socrates is a man $\supset$ Socrates is mortal' cannot logically imply 'There exists an entity p such that Socrates is a man $\supset$ p' unless the statement 'Socrates is mortal' in 'Socrates is a man $\supset$ Socrates is mortal' serves as a name of some entity or other.[6]

We may thus conclude that if a place-holder is to be quantified and hence function as a variable, its instances must be names.

The various entities which the instances of a variable designate in a given language L are technically called *the values of that variable in L*. The various entities which the instances of 'w', 'x', 'y', and 'z' designate in a given language L will therefore be called *the values of* 'w', 'x', 'y', *and* 'z' *in L*. These entities may be classified into two main groups:

(*a*) concrete entities like Athens, Socrates, and so on;
(*b*) abstract entities, that is:
 (*b*1) monadic properties like being an Athenian, being a Stoic, and so on;
 (*b*2) polyadic properties like being the birthplace of, being the son of, and so on;
 (*b*3) classes like the class of all Athenians, the class of all Stoics, and so on;
 (*b*4) relations like the relation birthplace of, the relation son of, and so on.

Properties usually serve as the values of 'F', 'G', and 'H', when these three letters are quantified; note, however, that they will also serve in a given language L as values of 'w', 'x', 'y', and 'z', if such property names as 'being an Athenian', 'being the birthplace of', and so on, are admitted in L among the instances of 'w', 'x', 'y', and 'z'. Classes and relations are usually treated as so-called *extensions* of monadic and polyadic predicates, respectively; note, however, that they will also serve in a given language L as values of 'w', 'x', 'y', and 'z', if such class and relation names as 'the class of all Athenians', 'the relation birthplace of', and so on, are admitted in L among the instances of 'w', 'x', 'y', and 'z'.[7]

The ontological status of all these entities has been heatedly discussed

[6]The entity in question is sometimes taken to be a fact, sometimes taken to be a proposition, the fact or proposition Socrates being mortal.

[7]We shall offer in chapter four a detailed analysis of classes and relations as extensions of predicates. The reader may, if he so wishes, enlarge (a)–(b4) to include facts or, even, propositions.

by philosophers, logicians, and mathematicians alike during the last two decades. One school, the so-called *Nominalist school,* acknowledges only concrete entities and bans from the vocabulary of science all arguments like 'being an Athenian', 'being the birthplace of', and so on, which purport to designate abstract entities of one sort or another. The so-called *Platonist school,* on the other hand, acknowledges both concrete and abstract entities. We shall remain neutral on this issue and let the reader select the instances of the four variables 'w', 'x', 'y', and 'z' according to his own taste.

We appointed above a fourth set of place-holders, the Greek letters 'φ', 'χ', 'ψ', 'ω', and 'τ', and used them in such contexts as 'For all φ', 'For some ψ', and so on; we may accordingly treat them as variables. The values of 'φ', 'χ', 'ψ', 'ω', and 'τ' will be the sentential or quantificational formulae which their instances designate. We are thus left with two sets of variables:

(a1) the five variables 'φ', 'χ', 'ψ', 'ω', and 'τ';
(a2) the four variables 'w', 'x', 'y', and 'z';

and two sets of dummies:

(b1) the four dummies 'p', 'q', 'r', and 's';
(b2) the three dummies 'F', 'G', and 'H'.

The variables listed in (a1) belong to metalogic, the variables listed in (a2) belong to logic. Both sets of variables play a critical role: they bridge the gap between a language (metalogic in the first case, logic in the other) and whatever universe of discourse that language may discourse about.[8]

16. THE CLASSICAL A-, E-, I-, AND O-FORMS

Among quantificational schemata, four call for special attention, namely:

1. $(x)(F(x) \supset G(x))$;
2. $(x)(F(x) \supset \sim G(x))$;
3. $(\mathrm{E}x)(F(x) \,.\, G(x))$;
4. $(\mathrm{E}x)(F(x) \,.\, \sim G(x))$.

If we substitute in these four schemata 'is a man' for 'F' and 'is fallible' for 'G', we obtain:

1a. For all x, x is a man $\supset$ x is fallible, i.e. All men are fallible;
2a. For all x, x is a man $\supset$ $\sim$(x is fallible), i.e. No man is fallible;

[8]In chapter three we shall introduce three additional sets of metalogical variables, to take as their values sentential dummies, predicate dummies, and argument variables, respectively.

3*a*. For some x, x is a man . x is fallible, i.e. Some men are fallible;

4*a*. For some x, x is a man . $\sim(x$ is fallible), i.e. Some men are not fallible.

1–4 thus yield, not all indeed, but the most current of all general statements. Names have been devised for them in Aristotelian logic: 1 being called an *A-schema*, 2 an *E-schema*, 3 an *I-schema*, and 4 an *O-schema*. These tags have been selected as follows: since schemata 1 and 3 show no denial sign, they are treated as affirmative schemata and, hence, assigned the first two vowels of the Latin verb '*Affirmo*'; since schemata 2 and 4 show denial signs, they are treated as negative schemata and, hence, assigned the two vowels of the Latin verb '*Nego*'. Instances of A-, E-, I-, and O-schemata have respectively been called *A-*, *E-*, *I-*, and *O-statements*.

The A-schema has many idiomatic readings. If 'is a man' is substituted for 'F' and 'is fallible' for 'G', then '$(x)(F(x) \supset G(x))$' may yield, besides 1*a*:

1. Every man is fallible;
2. Any man is fallible;
3. Men are fallible;
4. To be a man is to be fallible;
5. A man is fallible;
6. Man is fallible, and so on.

As shown by the equivalence of 'All men are fallible', 'Every man is fallible', and 'Any man is fallible', the three indefinite adjectives 'all', 'every', and 'any' are interchangeable in affirmative contexts. They are not interchangeable, however, in negative ones; 'I cannot beat every man on your team', for example, is not equivalent to 'I cannot beat any man on your team'. As shown by 'A man is fallible,' the indefinite article 'a' or 'an', though mostly used in particular statements, may also open universal ones. Similarly, the indefinite article 'the', though mostly used in singular statements, may also open universal ones like 'The ancient Greeks worshipped natural phenomena'.

The I-schema also has many idiomatic readings. If 'F' is instantiated as 'is a man' and 'G' as 'is illiterate', then '$(\mathrm{E}x)(F(x) \,.\, G(x))$' may yield:

1. Some men are illiterate;
2. There are men who are illiterate;
3. There are illiterate men;

and so on.

The second reading is of importance because relative clauses often turn up in everyday language; they should be read as factors of conjunctions.

'There are men who are illiterate' is thus paraphrased as '$(\mathrm{E}x)(x$ is a man . x is illiterate)', and 'All Americans who are Communists will be outlawed' paraphrased as '$(x)((x$ is an American . x is a Communist) $\supset$ x will be outlawed)'. As shown by the equivalence of 'There are men who are illiterate' and 'There are illiterate men', such relative clauses may condense into adjectives.

In Aristotelian logic, the four A-, E-, I-, and O-schemata were written as follows:

A: All F is G;
E: No F is G;
I: Some F is G;
O: Some F is not G.

This symbolism is somewhat misleading; it blurs the analogy between the A- and the E-schema and stresses unduly the part played by the verb 'is'. Aristotelian logic studied in detail the relations holding between A-, E-, I-, and O schemata; some of its results will be salvaged in the next section. At one point, though, Aristotelian logic and modern logic part company.

Aristotelian logic claimed that universal statements have a so-called *existential import*, that is, that instances of the two schemata '$(x)(F(x) \supset G(x))$' and '$(x)(F(x) \supset \sim G(x))$' are always asserted with the implicit understanding that the matching instance of '$(\mathrm{E}x)F(x)$' is true. This claim is discredited by such statements as 'All brakeless trains are dangerous', 'All brakeless cars are dangerous', and so on, which one might normally assert without thereby presupposing the existence of brakeless trains, brakeless cars, and the like. Modern logic accordingly denies existential import to all A- and E-schemata and rules that the existential clause '$(\mathrm{E}x)F(x)$', when implicit in the assertion of any A- or E-schema, be appended to the schema.

17. SELECTED QUANTIFICATIONAL SCHEMATA

We have seen above that certain sentential schemata like:

$$(p \,.\, q) \supset p,$$

have only true instances. Similarly, certain quantificational schemata like:

$$(x)F(x) \supset (\mathrm{E}x)F(x),$$

have only true instances. As we called the former schemata *sententially valid* and called their instances *sententially true*, so we shall call the latter schemata *quantificationally valid* and call their instances *quantificationally* true.

It follows from the definition of a quantificational truth and the definition of '$\supset$' that if a conditional $\ulcorner \varphi \supset \psi \urcorner$ is quantificationally true, then its antecedent cannot be true without its consequent being also true; we may accordingly say of $\ulcorner \varphi \supset \psi \urcorner$ that its antecedent *logically* (or, if we wish to be more specific, *quantificationally*) *implies* its consequent. It follows likewise from the definition of a quantificational truth and the definition of '$\equiv$' that if a biconditional $\ulcorner \varphi \equiv \psi \urcorner$ is quantificationally true, then its left-hand component cannot be true without its right-hand component being also true, and *vice-versa*; we may accordingly say of $\ulcorner \varphi \equiv \psi \urcorner$ that its two components are *logically* (or, if we wish to be more specific, *quantificationally*) *equivalent.*

Quantificationally valid schemata may be of two kinds: some are tautologies, others are not. To illustrate this point, let us consider the two valid schemata:

$$(x)F(x) \supset (x)F(x)$$

and

$$(x)F(x) \supset (\mathrm{E}x)F(x),$$

and subject them to the truth-table test. We get in the first case:

$(x)F(x)$	$(x)F(x) \supset (x)F(x)$
T	T
F	T

We get in the second case:

$(x)F(x)$	$(\mathrm{E}x)F(x)$	$(x)F(x) \supset (\mathrm{E}x)F(x)$
T	T	T
F	T	T
T	F	F
F	F	T

The first schema is thus a tautology; the second, however, is not.

We have seen above that certain sentential schemata like:

$$p \,.\, \sim p,$$

have only false instances. Similarly, certain quantificational schemata like:

$$(\mathrm{E}x)F(x) \equiv (x)\sim F(x),$$

have only false instances. As we called the former *sententially contravalid schemata,* so we shall call the latter *quantificationally contravalid schemata.*

Quantificationally contravalid schemata may be of two kinds: some are contradictions, others are not. To illustrate this point, let us consider the two contravalid schemata:

$$(x)F(x) \equiv \sim(x)F(x)$$

and

$$(\mathrm{E}x)(F(x) \equiv (x)\sim F(x),$$

and subject them to the truth-table test. We get in the first case:

$(x)F(x)$	$(x)F(x) \equiv \sim(x)F(x)$
T	F F
F	F T

We get in the second case:

$(\mathrm{E}x)F(x)$	$(x)\sim F(x)$	$(\mathrm{E}x)F(x) \equiv (x)\sim F(x)$
T	T	T
F	T	F
T	F	F
F	F	T

The first schema is thus a contradiction; the second, however, is not.

We have seen above that certain sentential schemata like:

$$p \cdot q,$$

have both true and false instances. Similarly, certain quantificational schemata like:

$$(\mathrm{E}x)F(x) \;.\; (\mathrm{E}x)\sim F(x),$$

have both true and false instances. As we called the former *sententially indeterminate schemata*, so we shall call the latter *quantificationally indeterminate schemata*.

We may insert here two remarks on the testing of quantificational schemata by truth-tables:

1. A quantified schema always counts as one sentential unit, whether that schema be atomic or molecular. In '$((x)(F(x) \supset G(x)) \;.\; (x)(G(x) \supset H(x))) \supset (x)(F(x) \supset H(x))$', for example, '$(x)(F(x) \supset G(x))$', '$(x)(G(x) \supset H(x))$', and '$(x)(F(x) \supset H(x))$' must each count as one sentential unit, even though each one is the closure of a molecular schema.

2. When a quantified schema counts as one sentential unit, any part of it must count as a second sentential unit. In '$(x)F(x) \supset (F(x) \supset$

$(y)G(y))$', for example, '$(x)F(x)$' and '$F(x)$' must count as two sentential units.

We now list and label 42 quantificationally valid schemata. The first four, called *laws of Opposition,* correlate universal and existential quantifiers. Q30c and Q30d are especially important; they show that a universal quantifier can be written as an existential one flanked by denial signs and, conversely, that an existential quantifier can be written as a universal one flanked by denial signs. The reader can check this on the following two pairs of statements: 'Everything is black' and 'It is not the case that something is not black', 'Something is black' and 'It is not the case that nothing is black'. The two members of each pair are easily seen to be logically equivalent.

Q30a: $\sim(x)F(x) \equiv (\mathrm{E}x)\sim F(x)$;
Q30b: $\sim(\mathrm{E}x)F(x) \equiv (x)\sim F(x)$;
Q30c: $(x)F(x) \equiv \sim(\mathrm{E}x)\sim F(x)$;
Q30d: $(\mathrm{E}x)F(x) \equiv \sim(x)\sim F(x)$.

Our next four laws, also called *laws of Opposition*, correlate A-, E-, I-, and O-schemata with O-, I-, E-, and A-schemata; they originated with Aristotle.

Q31a: $(x)(F(x) \supset G(x)) \equiv \sim(\mathrm{E}x)(F(x) \,.\, \sim G(x))$;
Q31b: $(x)(F(x) \supset \sim G(x)) \equiv \sim(\mathrm{E}x)(F(x) \,.\, G(x))$;
Q31c: $(\mathrm{E}x)(F(x) \,.\, G(x)) \equiv \sim(x)(F(x) \supset \sim G(x))$;
Q31d: $(\mathrm{E}x)(F(x) \,.\, G(x)) \equiv \sim(x)(F(x) \supset G(x))$.

The reader can intuitively check their validity on the following four pairs of statements:

1. 'All men are mortal' and 'It is not the case that some men are not mortal';
2. 'No man is infallible' and 'It is not the case that some men are infallible';
3. 'Some men are fallible' and 'It is not the case that no man is infallible';
4. 'Some men are not wise' and 'It is not the case that all men are wise'.

The next law, due again to Aristotle, is called *the law of the Categorical Syllogism*; two versions of it are listed, to be respectively identified as *Barbara* and *Darii.*

Q32a: $((x)(G(x) \supset H(x)) \,.\, (x)(F(x) \supset G(x))) \supset (x)(F(x) \supset H(x))$;
Q32b: $((x)(G(x) \supset H(x)) \,.\, (\mathrm{E}x)(F(x) \,.\, G(x))) \supset (\mathrm{E}x)(F(x) \,.\, H(x))$.

Among instances of Q32a we may quote:

(All men are mortal . All Americans are men) $\supset$ All Americans are mortal;

and among the instances of Q32b:

(All negroes are race conscious. Some Americans are negroes) $\supset$ Some Americans are race conscious.[9]

Syllogisms were the main topic of Aristotle's *Organon*; we shall therefore study them in some detail. As shown by *Barbara* and *Darii*, a syllogism is a conditional made up of three quantified schemata, two of which serve as its antecedent and one as its consequent. Each schema is made up in turn of two predicate dummies. Of the two predicate dummies entering the consequent, the right one, 'H', is called the *major term*; the left one, 'F', is called the *minor term*. Both 'H' and 'F' enter one factor of the antecedent, 'H' the first or so-called *major factor*, 'F' the second or so-called *minor factor*. A third predicate dummy, 'G', enters both the major and the minor schemata; it is called the *middle term*.

Barbara may this be schematized as follows:

$$((G \supset H) \,.\, (F \supset G)) \supset (F \supset H);$$

Darii, in turn, may be schematized as follows:

$$((G \supset H) \,.\, (F \,.\, G)) \supset (F \,.\, H).$$

Both syllogisms assign to the three predicate dummies 'F', 'G', and 'H' similar positions, namely:

$$((G \quad H) \,.\, (F \quad G)) \supset (F \quad H).$$

This pattern is usually called a *figure*. If we permute 'F', 'G', and 'H' in the major and the minor schemata, we obtain a total of 4 figures, namely:

First figure: $((G \quad H) \,.\, (F \quad G)) \supset (F \quad H)$;
Second figure: $((H \quad G) \,.\, (F \quad G)) \supset (F \quad H)$;
Third figure: $((G \quad H) \,.\, (G \quad F)) \supset (F \quad H)$;
Fourth figure: $((H \quad G) \,.\, (G \quad F)) \supset (F \quad H)$.

As the reader may check, *Barbara* and *Darii* are two syllogisms of the first figure. The following two syllogisms, respectively labelled *Celarent* and *Ferio*,

[9]The famous conditional:

(All men are mortal . Socrates is a man) $\supset$ Socrates is mortal,

is not an instance of *Barbara*, but of the schema '$((x)(F(x) \supset G(x)) \,.\, F(y)) \supset G(y)$'; it is therefore not a syllogism, however often it may have been quoted as such.

$$((x)(G(x) \supset \sim H(x)) \,.\, (x)(F(x) \supset G(x))) \supset (x)(F(x) \supset \sim H(x)),$$
$$((x)(G(x) \supset \sim H(x)) \,.\, (\mathrm{E}x)(F(x) \,.\, G(x))) \supset (\mathrm{E}x)(F(x) \,.\, \sim H(x)),$$

also belong to the first figure. Of the four syllogisms in question,

the first, *Barbara*, follows the pattern: '(A . A) ⊃ A';
the second, *Darii*, follows the pattern: '(A . I) ⊃ I';
the third, *Celarent*, follows the pattern: '(E . A) ⊃ E';
the fourth, *Ferio*, follows the pattern: '(E . I) ⊃ O'.

These patterns or *moods*, as the Aristotelians called them, are only 4 out of 64 possible ones. Given the 4 schemata A, E, I, and O, we may set up 16 two-factored antecedents:

A . A	E . A	I . A	O . A
A . E	E . E	I . E	O . E
A . I	E . I	I . I	O . I
A . O	E . O	I . O	O . O

If we successively assign to each one of these 16 antecedents an A-, E-, I-, and O-consequent, we obtain a total of 64 moods within each figure and hence a total of 256 syllogisms.

Only 15 of these syllogisms are valid, however. The Aristotelian tradition assigned them names which identify their mood through use of the key letters 'A', 'E', 'I', and 'O'; they read:

First figure: *Barbara*, *Celarent*, *Darii*, and *Ferio*;
Second figure: *Cesare*, *Camestres*, *Festino*, and *Baroco*;
Third figure: *Datisi*, *Feriso*, *Disamis*, and *Bocardo*;
Fourth figure: *Calemes*, *Fresison*, and *Dimatis*.

We may remark that the fourth figure was not officially acknowledged by Aristotle; tradition credited its discovery to the famous Greek physician Galen (about 200 A. D.), but mistakenly so, it now appears.

Nine more syllogisms can be shown to be valid if their antecedent is supplemented with an existential clause, namely:

First figure:	1. (A . A) ⊃ I;
	2. (E . A) ⊃ O;
Second figure:	3. (A . E) ⊃ O;
	4. (E . A) ⊃ O;
Third figure:	5. (A . A) ⊃ I;
	6. (E . A) ⊃ O;
Fourth figure:	7. (A . E) ⊃ O;
	8. (E . A) ⊃ O;
	9. (A . A) ⊃ I.

The existential clause required is:

'$(\mathrm{E}x)F(x)$' in the first, second, third, fourth, and seventh cases;
'$(\mathrm{E}x)G(x)$' in the fifth, sixth, and eighth cases;
'$(\mathrm{E}x)H(x)$' in the ninth case.

For reasons of its own, tradition endorsed only four of these nine syllogisms, namely: the fifth, sixth, eighth, and ninth, and labelled them: *Darapti*, *Felapton*, *Fesapo*, and *Bamalip*. It thus ended up with a set of 19, rather than 24, valid syllogisms; the former figure is still a liberal one, though, in view of the minor part played in logic by the syllogism.

Before resuming our list of valid quantificational schemata, let us introduce a technical phrase, the phrase 'is true of'. We shall say that a predicate '—— ——' is true of everything, true of something, or true of a given thing a if the statement '(x)(—— x ——)', the statement '$(\mathrm{E}x)$(—— x ——)', or the statement '—— a ——', respectively, is true. The predicate 'is self-identical', for example, is true of everything, the predicate 'is a Belgian city south of Brussels' true of something, and the predicate 'is black' true of coal.

Q33: $(x)F(x) \supset F(y)$.

According to Q33, called *the law of Specification*, a predicate, if true of everything, is true of any given thing. Among the instances of Q33, let us quote:

$$(x)(x = x) \supset \text{Socrates} = \text{Socrates}.$$

Q34: $F(y) \supset (\mathrm{E}x)F(x)$.

According to Q34, called *the law of Particularization*, a predicate, if true of a given thing, is true of something. Among the instances of Q34, let us quote:

$$\text{Socrates is a man} \supset (\mathrm{E}x)(x \text{ is a man}).$$

Q33 allows for the deduction of singular statements from universal ones, Q34 the deduction of particular statements from singular ones.

The next law, called *the law of Subalternation*, shows that a predicate, if true of everything, must be true of something; it thus allows for the deduction of particular statements from universal ones.

Q35a: $(x)F(x) \supset (\mathrm{E}x)F(x)$.

Among the instances of Q35a falls the conditional:

$$\text{Everything is black} \supset \text{Something is black}.$$

Appended to Q35a is a schema allowing for the deduction of I-statements from A-statements:

Q35b: $((x)(F(x) \supset G(x)) \,.\, (\mathrm{E}x)F(x)) \supset (\mathrm{E}x)(F(x) \,.\, G(x))$.

The existential clause '$(\mathrm{E}x)F(x)$' is essential to the validity of Q35b; '$(x)(F(x) \supset G(x)) \supset (\mathrm{E}x)(F(x) \,.\, G(x))$' admits indeed false instances such as:

All Centaurs run $\supset$ There are Centaurs that run.

The distinction between Q35a and Q35b should be carefully noted. Whereas Q35b allows for the deduction of '$(\mathrm{E}x)$(x is a Centaur . x runs)' from the two premises '(x)(x is a Centaur $\supset$ x runs)' and '$(\mathrm{E}x)$(x is a Centaur)', Q35a allows only for the deduction of '$(\mathrm{E}x)$(x is a Centaur $\supset$ x runs)' from the premise '(x)(x is a Centaur $\supset$ x runs)'. The shift from the existential conditional: '$(\mathrm{E}x)$(x is a Centaur $\supset$ x runs)', to the existential conjunction: '$(\mathrm{E}x)$(x is a Centaur . x runs)', calls for the extra premise: '$(\mathrm{E}x)$(x is a Centaur)'.

The next two laws show that a universal quantifier can be distributed through a conjunction and an existential quantifier through an alternation:

Q36a: $(x)(F(x) \,.\, G(x)) \equiv ((x)F(x) \,.\, (x)G(x))$;
Q36b: $(\mathrm{E}x)(F(x) \vee G(x)) \equiv ((\mathrm{E}x)F(x) \vee (\mathrm{E}x)G(x))$.

A variant of Q36b is appended:

Q36c: $(\mathrm{E}x)(F(x) \supset G(x)) \equiv ((x)F(x) \supset (\mathrm{E}x)G(x))$.

Q36a and Q36b are two of the formal laws, mentioned on page 54, which govern the translation of non-sentential connectives into sentential ones within general statements. According to Q36a, for example,

Everything is finite in space and time

is logically equivalent to:

Everything is finite in space . Everything is finite in time.

According to Q36b,

Something is jammed or broken

is logically equivalent to:

Something is jammed $\vee$ Something is broken.

In view of Q36a and Q36b, one might expect that quantifiers could always be distributed through molecular formulae; this, however, is not

the case. Whereas the two statements: 'Something is tall' and 'Something is short', are true, the statement: 'Something is both tall and short', is false. The following biconditional is therefore not valid:

$$(\mathrm{E}x)(F(x) \,.\, G(x)) \equiv ((\mathrm{E}x)F(x) \,.\, (\mathrm{E}x)G(x)).$$

Half of it is, however:

$$(\mathrm{E}x)(F(x) \,.\, G(x)) \supset ((\mathrm{E}x)F(x) \,.\, (\mathrm{E}x)G(x));$$

if, for example, something is white and soft, then something is white and something is soft. There is a large number of *laws of Distributivity* which hold only as conditionals; some are recorded here.

Q36d: $(x)(F(x) \supset G(x)) \supset ((x)(Fx) \supset (x)G(x))$;
Q36e: $(x)(F(x) \equiv G(x)) \supset ((x)F(x) \equiv (x)G(x))$;
Q36f: $(\mathrm{E}x)(F(x) \,.\, G(x)) \supset ((\mathrm{E}x)F(x) \,.\, (\mathrm{E}x)G(x))$;
Q36g: $((x)F(x) \vee (x)G(x)) \supset (x)(F(x) \vee G(x))$;
Q36h: $(x)(F(x) \supset G(x)) \supset ((\mathrm{E}x)F(x) \supset (\mathrm{E}x)G(x))$;
Q36i: $((\mathrm{E}x)F(x) \supset (\mathrm{E}x)G(x)) \supset (\mathrm{E}x)(F(x) \supset G(x))$.

The next laws to be recorded are *laws of Confinement.* A quantifier, say '(x)' or '$(\mathrm{E}x)$', may sometimes govern a molecular schema without the argument variable 'x' itself occurring in each component of the schema. Such is the case, for example, with '$(\mathrm{E}x)$' in

$$(\mathrm{E}x)(F(x) \,.\, p).$$

The law of Confinement tells us that '$(\mathrm{E}x)$' may be shifted in front of '$F(x)$' and that '$(\mathrm{E}x)(F(x) \,.\, p)$' may accordingly be written as:

$$(\mathrm{E}x)F(x) \,.\, p.$$

To check this fact, let us instantiate '$F(x)$' as 'x is a Hellenist and x likes Pindar' and 'p' as 'Pindar is a truly great poet'. '$(\mathrm{E}x)(F(x) \,.\, p)$' will then yield: 'Some Hellenists like Pindar who is a truly great poet'; '$(\mathrm{E}x)F(x).\ p$', on the other hand, will yield: 'Some Hellenists like Pindar and Pindar is a truly great poet'. The two statements are easily seen to be logically equivalent. Twelve cases of confinement are listed below:

Q37a: $(x)(p \,.\, F(x)) \equiv (p \,.\, (x)F(x))$;
Q37b: $(x)(F(x) \,.\, p) \equiv ((x)F(x) \,.\, p)$;
Q37c: $(\mathrm{E}x)(p \,.\, F(x)) \equiv (p \,.\, (\mathrm{E}x)F(x))$;
Q37d: $(\mathrm{E}x)(F(x) \,.\, p) \equiv ((\mathrm{E}x)F(x) \,.\, p)$;
Q37e: $(x)(p \vee F(x)) \equiv (p \vee (x)F(x))$;
Q37f: $(x)(F(x) \vee p) \equiv ((x)F(x) \vee p)$;
Q37g: $(\mathrm{E}x)(p \vee F(x)) \equiv (p \vee (\mathrm{E}x)F(x))$;

Q37h: $(\mathrm{E}x)(F(x) \vee p) \equiv ((\mathrm{E}x)F(x) \vee p)$;
Q37i: $(x)(p \supset F(x)) \equiv (p \supset (x)F(x))$;
Q37j: $(x)(F(x) \supset p) \equiv ((\mathrm{E}x)F(x) \supset p)$;
Q37k: $(\mathrm{E}x)(p \supset F(x)) \equiv (p \supset (\mathrm{E}x)F(x))$;
Q37l: $(\mathrm{E}x)(F(x) \supset p) \equiv ((x)F(x) \supset p)$.

Of these laws only Q37j and Q37l call for comment: they show that a universal quantifier, when switched from a conditional to its antecedent, must be rewritten as an existential one, and that an existential quantifier, when similarly switched from a conditional to its antecedent, must be rewritten as a universal one. It is not easy to find instances of this shift in everyday language. Let us remark, however, that '(x)(x is broken $\supset$ x will have to be repaired)' may be read: 'If anything is broken, then it will have to be repaired'. Using this cue, we may instantiate '$(x)(F(x) \supset p)$' as 'If anything is broken, I shall get hell', where 'is broken' is substituted for 'F' and 'I shall get hell' for 'p'. Through the same substitutions '$(\mathrm{E}x)F(x) \supset p$' yields: 'If something is broken, I shall get hell'. But the two statements: 'If anything is broken, I shall get hell' and 'If something is broken, I shall get hell', are logically equivalent. Hence the intuitive validity of the schema:

$$(x)(F(x) \supset p) \equiv ((\mathrm{E}x)F(x) \supset p).$$

The previous schemata showed only one-place or *monadic* predicate dummies. The next schemata will, on the other hand, show two-place or *dyadic* predicate dummies. Before stating them, we shall give a few cues as to the translation of everyday statements into n-adic ($n > 1$) or *polyadic* quantificational idiom.

1. Let us first take statements showing a polyadic predicate but only one quantifier, like:

Every Bostonian liked Curley (1)

and

Some Bostonians liked Curley (2).

If we abbreviate 'x is a Bostonian' as '$\mathrm{B}(x)$', 'x liked y' as '$\mathrm{L}(x,y)$', and 'Curley' as 'c', then we may translate (1) into:

$$(x)(\mathrm{B}(x) \supset \mathrm{L}(x,\mathrm{c})),$$

and (2) into:

$$(\mathrm{E}x)(\mathrm{B}(x) \,.\, \mathrm{L}(x,\mathrm{c})).$$

2. Let us next take statements showing two quantifiers, like:

Every Bostonian snubs every New Yorker (3)

and

$$\text{Some Bostonians snub some New Yorkers} \qquad (4).$$

If we abbreviate 'x is a Bostonian' as '$B(x)$', 'x is a New Yorker' as '$N(x)$', and 'x snubs y' as '$S(x,y)$', then we may translate (3) into:

$$(x)(B(x) \supset (y)(N(y) \supset S(x,y))) \qquad (5),$$

and (4) into:

$$(Ex)(B(x) \,.\, (Ey)(N(y) \,.\, S(x,y))).$$

To translate (3) into (5) we may proceed as follows:

1. The statement: 'Every Bostonian snubs every New Yorker', is, like the statement: 'Every Bostonian liked Curley', a statement about Bostonians. We may thus open the translation of (3) with the universal quantifier '(x)' and write down the clause: 'x is a Bostonian $\supset$', to restrict the values of 'x' to Bostonians; this operation yields:

$$(x)(x \text{ is a Bostonian } \supset x \text{ snubs every New Yorker}).$$

2. In order to translate the clause: 'x snubs every New Yorker', we may turn it into the more familiar: 'Every New Yorker is snubbed by x', that is:

$$(y)(y \text{ is a New Yorker } \supset y \text{ is snubbed by } x).$$

But 'y is snubbed by x' is equivalent to 'x snubs y'; we may therefore interchange them in our conditional.

3. Combining these two results, we get:

$$(x)(x \text{ is a Bostonian } \supset (y)(y \text{ is a New Yorker } \supset x \text{ snubs } y)),$$

that is:

$$(x)(B(x) \supset (y)(N(y) \supset S(x,y))).$$

Another example may be worked out where, this time, the phrase 'every' is used in conjunction with the phrase 'some', namely:

Everyone on my team outplays someone on your team.

This statement may be analyzed as follows:

$$(x)(x \text{ is on my team } \supset x \text{ outplays someone on your team}).$$

But 'x outplays someone on your team' is equivalent to 'someone on your team is outplayed by x', that is:

$$(Ey)(y \text{ is on your team } .\; y \text{ is outplayed by } x).$$

But 'y is outplayed by x' is equivalent to 'x outplays y'. Our two clauses may thus combine into:

$$(x)(x \text{ is on my team} \supset (\mathrm{E}y)(y \text{ is on your team} \,.\, x \text{ outplays } y)).$$

If we abbreviate 'x is on my team' as 'M(x)', 'y is on your team' as 'Y(y)', and 'x outplays y' as 'O(x,y)', then we may phrase our original statement as follows:

$$(x)(\mathrm{M}(x) \supset (\mathrm{E}y)(\mathrm{Y}(y) \,.\, \mathrm{O}(x,y))).$$

Let us note in passing that everyday language often uses the two pronouns 'the former' and 'the latter' in connection with dyadic predicates. 'If a person despises another person, then the latter invariably hates the former', for example, translates into:

$$(x)(y)(\mathrm{D}(x,y) \supset \mathrm{H}(y,x)),$$

where 'D(x,y)' is short for 'x despises y' and 'H(y,x)' is short for 'x hates y'. Let us also note that when a predicate is followed in a statement φ by one of the two pronouns 'one another' and 'each other', then it has to appear twice in the symbolic translation of φ. 'Bostonians and New Yorkers snub each other', for example, becomes:

$$(x)(y)((\mathrm{B}(x) \,.\, \mathrm{N}(y)) \supset (\mathrm{S}(x,y) \,.\, \mathrm{S}(y,x))).$$

When, however, the predicate in question is symmetrical, there is no need repeating it. 'All masterpieces are contemporary', for example, may expand into:

$$(x)(y)((x \text{ is a masterpiece} \,.\, y \text{ is a masterpiece}) \supset x \text{ is contemporary with } y);$$

the further clause '. y is contemporary with x' would clearly be useless here.

This last example is of interest because it shows that the pronouns 'each other' and 'one another' are often dropped from contexts where they formally belong; 'All masterpieces are contemporary', for example, should formally read: 'All masterpieces are each other's contemporaries'. Sometimes, also, these two pronouns are replaced by adverbs like 'mutually'. 'All intimate friends are mutually critical', for example, should expand into:

$$(x)(y)((x \text{ is an intimate friend of } y \,.\, y \text{ is an intimate friend of } x) \supset (x \text{ is critical of } y \,.\, y \text{ is critical of } x)).$$

The laws of polyadic quantification promised above are seven in number. The first four are dyadic analogues of Q30a–Q30d; we may accordingly call them *laws of dyadic Opposition.*

Q38a: $\sim(x)(y)F(x,y) \equiv (\mathrm{E}x)(\mathrm{E}y)\sim F(x,y)$;
Q38b: $\sim(\mathrm{E}x)(\mathrm{E}y)F(x,y) \equiv (x)(y)\sim F(x,y)$;
Q38c: $(x)(y)F(x,y) \equiv \sim(\mathrm{E}x)(\mathrm{E}y)\sim F(x,y)$;
Q38d: $(\mathrm{E}x)(\mathrm{E}y)F(x,y) \equiv \sim(x)(y)\sim F(x,y)$.

The last three are called *laws of Permutation*; the first two show that two universal or two existential quantifiers may be permuted at will; the third one shows that a some-all statement logically implies an all-some one.

Q39a: $(x)(y)F(x,y) \equiv (y)(x)F(x,y)$;
Q39b: $(\mathrm{E}x)(\mathrm{E}y)F(x,y) \equiv (\mathrm{E}y)(\mathrm{E}x)F(x,y)$;
Q39c: $(\mathrm{E}x)(y)F(x,y) \supset (y)(\mathrm{E}x)F(x,y)$.

The converse of Q39c is obviously not valid. '$(y)(\mathrm{E}x)(y = x)$', that is, 'Everything is identical with something or other', is true; yet '$(\mathrm{E}x)(y)(y = x)$', that is, 'There is something with which everything is identical', may be false.

18. RULES OF QUANTIFICATIONAL DEDUCTION

The deductive techniques learned in chapter one provided for all sentential deductions. Do they also provide for all quantificational ones? The answer, clearly, is: "No." The following conditional, for example:

(All men are mortal . All Americans are men) ⊃ All Americans are mortal,

is logically true, it being an instance of Q32a:

$$((x)(G(x) \supset H(x)) \,.\, (x)(F(x) \supset G(x))) \supset (x)(F(x) \supset H(x)).$$

We may thus conclude that the two premises:

P1: All men are mortal (1),
P2: All Americans are men (2),

logically imply the conclusion:

C1: All Americans are mortal (3).

Yet this conclusion is not deducible from its premises through R1–R4. Adjunction can lead us indeed from (1) and (2) to

$$(1) \,.\, (2);$$

this, however, is as close as we shall get to our conclusion through sentential means. We thus have to adopt further rules of deduction. Before doing so, however, we shall modify R3, the rule of Sentential Insertion.

R3, in its original form, enabled us to supplement the premises of a deduction with extra statements; these statements resulted from the substitution of statements for the letters of recorded sentential schemata. R3, in its revised form, will enable us to supplement the premises of a deduction with quasi-statements as well as statements; these quasi-

statements will result from the substitution of quasi-statements and, possibly, statements for the letters of recorded sentential schemata. If we agree to call a statement an *instance* and a quasi-statement a *quasi-instance* of a schema, then we may modify R3 to read:

R3: "If φ is an instance or a quasi-instance of a recorded sentential schema, then φ may serve as a premise in any quantificational deduction."

The substitution process whereby instances and quasi-instances arise from recorded sentential schemata will be studied below.

R1, R2, and R4 may be kept as originally stated; but as R3 now allows for the insertion of statements and quasi-statements, so R1, R2, and R4 will allow for the detachment, adjunction, and interchange of statements and quasi-statements. The impact of this revision will soon be felt.

Our new rules are three in number: *Quantificational Insertion* (R5), *Relettering* (R6), and *Universalization* (R7). R5 plays in connection with recorded quantificational schemata the part R3 plays in connection with recorded sentential schemata; it may accordingly read:

R5: "If φ is an instance or a quasi-instance of a recorded quantificational schema, then φ may serve as a premise in any quantificational deduction."

The substitution process whereby instances or quasi-instances arise from recorded quantificational schemata will be studied below.

With R5 at hand, we can deduce (3) above from (1) and (2). We need only supplement our two premises with the following instance of Q32a:

(All men are mortal . All Americans are men) ⊃ All Americans are mortal (4).

Adjunction leads us again from (1) and (2) to

$$(1) \,.\, (2) \tag{5}$$

and Detachment leads us from (4) and (5) to (3), our conclusion.

If we abbreviate 'x is a man' as 'M(x)', 'x is an American' as 'A(x)', and 'x is mortal' as 'N(x)', then we may roughly record our deduction as follows:

P1:	$(x)(\mathrm{M}(x) \supset \mathrm{N}(x))$	
P2:	$(x)(\mathrm{A}(x) \supset \mathrm{M}(x))$	
P3:	$((x)(\mathrm{M}(x) \supset \mathrm{N}(x)) \,.\, (x)(\mathrm{A}(x) \supset \mathrm{M}(x)) \supset (x)(\mathrm{A}(x) \supset \mathrm{N}(x))$	Ins. Q32a
4:	$(x)(\mathrm{M}(x) \supset \mathrm{N}(x)) \,.\, (x)(\mathrm{A}(x) \supset \mathrm{M}(x))$	Adj 1 and 2
Cl:	$(x)(\mathrm{A}(x) \supset \mathrm{N}(x))$	Det 3 and 4

The following deduction can also be carried out through Quantificational Insertion; we are given the three premises:

P1: All men are mortal (6),
P2: All Americans are men (7),
P3: There are Americans (8),

and asked to deduce from them the conclusion:

Cl: Some Americans are mortal (9).

The conditional:

$$((6) \,.\, (7) \,.\, (8)) \supset (9),$$

does not match, as '((1) . (2)) ⊃ (3)' did, any of the schemata recorded in section 17. Quantificational truths are easily found, however, matching recorded schemata and allowing for the deduction of (9) from (6)–(8) through familiar channels, namely:

(All men are mortal . All Americans are men) ⊃ All Americans are mortal (10),

an instance of Q32a, and

(All Americans are mortal . There are Americans) ⊃ Some Americans are mortal (11),

an instance of Q35b.

The deduction may run as follows:

1. From (6) and (7) deduce:

$$(6) \,.\, (7) \tag{12}$$

by Adjunction.

2. From (10) and (12) deduce:

All Americans are mortal (13),

by Detachment.

3. From (13) and (8) deduce:

$$(13) \,.\, (8) \tag{14}$$

by Adjunction.

4. From (11) and (14) deduce conclusion (9) by Detachment.

Before stating R6, let us introduce an auxiliary concept, the concept of a relettering. By *a relettering of a formula* φ we shall understand any result of substituting a variable foreign to φ for all the occurrences of a variable bound in φ by a given occurrence of a quantifier. If, for instance,

φ is '$(x)(x = x) \supset (x)(x = x)$', then '$(y)(y = y) \supset (x)(x = x)$', the result of substituting 'y' for all the occurrences of 'x' bound in φ by the first occurrence of '(x)', and '$(x)(x = x) \supset (y)(y = y)$', the result of substituting 'y' for all the occurrences of 'x' bound in φ by the second occurrence of '(x)', will both count as reletterings of φ. Similarly, if φ is '$(x)(y)(x = y)$', then '$(w)(y)(w = y)$', the result of substituting 'w' for all the occurrences of 'x' bound in φ by '(x)', and '$(x)(w)(x = w)$', the result of substituting 'w' for all the occurrences of 'y' bound in φ by '(y)', will both count as reletterings of φ.

With this definition at hand, we are now ready to state R6, *the rule of Relettering*:

R6: "From φ one may deduce ψ, if ψ is a relettering of φ."

R6 may be used to deduce from a statement φ a relettering ψ thereof, or to deduce from a quasi-statement φ a relettering ψ thereof. To illustrate the first possibility, let us translate statements (1)–(3) above into:

P1: $(y)(\mathrm{M}(y) \supset \mathrm{N}(y))$ (15),
P2: $(y)(\mathrm{A}(y) \supset \mathrm{M}(y))$ (16),
Cl: $(y)(\mathrm{A}(y) \supset \mathrm{N}(y))$ (17),

rather than:

P1: $(x)(\mathrm{M}(x) \supset \mathrm{N}(x))$,
P2: $(x)(\mathrm{A}(x) \supset \mathrm{M}(x))$,
Cl: $(x)(\mathrm{A}(x) \supset \mathrm{N}(x))$.

To deduce (17) from (15) and (16) we may adopt either one of the following two strategies:

Strategy 1: Deduce '$(x)(\mathrm{M}(x) \supset \mathrm{N}(x))$' from (15) by R6, '$(x)(\mathrm{A}(x) \supset \mathrm{M}(x))$' from (16) by R6 again, deduce '$(x)(\mathrm{A}(x) \supset \mathrm{N}(x))$' from '$(x)(\mathrm{M}(x) \supset \mathrm{N}(x))$' and '$(x)(\mathrm{A}(x) \supset \mathrm{M}(x))$' as above, and finally deduce (17) from '$(x)(\mathrm{A}(x) \supset \mathrm{N}(x))$' by R6.

Strategy 2: Assume the following instance of Q32a by R5:

$$((x)(\mathrm{M}(x) \supset \mathrm{N}(x)) \,.\, (x)(\mathrm{A}(x) \supset \mathrm{M}(x))) \supset (x)(\mathrm{A}(x) \supset \mathrm{N}(x)) \quad (18);$$

deduce from (18) the relettering thereof:

$$((y)(\mathrm{M}(y) \supset \mathrm{N}(y)) \,.\, (x)(\mathrm{A}(x) \supset \mathrm{M}(x))) \supset (x)(\mathrm{A}(x) \supset \mathrm{N}(x)) \quad (19),$$

by R6;

deduce from (19) the relettering thereof:

$$((y)(\mathrm{M}(y) \supset \mathrm{N}(y)) \,.\, (y)(\mathrm{A}(y) \supset \mathrm{M}(y))) \supset (x)(\mathrm{A}(x) \supset \mathrm{N}(x)) \quad (20),$$

by R6 again;

deduce from (20) the relettering thereof:

$$((y)(\mathrm{M}(y) \supset \mathrm{N}(y)) \,.\, (y)(\mathrm{A}(y) \supset \mathrm{M}(y))) \supset (y)(\mathrm{A}(y) \supset \mathrm{N}(y)) \qquad (21),$$

by R6 again;

deduce from (15) and (16):

$$(15) \,.\, (16) \qquad (22),$$

by Adjunction;

deduce (17) from (21) and (22) by Detachment.

Since Q30a-Q37l are written with an 'x', it is wise to stick by 'x' when translating (1)–(3) or any other set of monadic formulae into quantificational idiom, and dispense thereby with R6. When dealing with polyadic formulae, however, we must use other variables besides 'x' and hence may come to need R6. A case in point is the following which also illustrates the deduction from quasi-statements of reletterings thereof.

Let us assume the single premise:

P1: No one is the father of his own father,

and deduce from it the conclusion:

Cl: No one is his own father.

Abbreviating 'is the father of' as 'F', we shall translate our premise and conclusion into:

$$(x)(w)(\mathrm{F}(w,x) \supset {\sim}\mathrm{F}(x,w)) \qquad (23)$$

and

$$(y){\sim}\mathrm{F}(y,y) \qquad (24).$$

Let us temporarily delete the '(y)' which opens (24) and work on the quasi-statement:

$${\sim}\mathrm{F}(y,y) \qquad (25).$$

To deduce (25) from (23) we could argue as follows:

1. Since, by (23), '$(w)(\mathrm{F}(w,\ \ ) \supset {\sim}\mathrm{F}(\ \ ,w))$' is true of everything, it must be true of y; hence

$$(w)(\mathrm{F}(w,y) \supset {\sim}\mathrm{F}(y,w)) \qquad (26).$$

2. Since, by (26), '$\mathrm{F}(\ \ ,y) \supset {\sim}\mathrm{F}(y,\ \ )$' is true of everything, it must be true again of y; hence

$$\mathrm{F}(y,y) \supset {\sim}\mathrm{F}(y,y) \qquad (27).$$

3. But, by T18, (27) is logically equivalent to '${\sim}\mathrm{F}(y,y) \vee {\sim}\mathrm{F}(y,y)$', which, by T4b, is logically equivalent to '${\sim}\mathrm{F}(y,y)$', our tentative conclusion.

The quasi-statement:

$$(x)(w)(\mathrm{F}(w,x) \supset \sim\mathrm{F}(x,w)) \supset (w)(\mathrm{F}(w,y) \supset \sim\mathrm{F}(y,w)),$$

needed in 1, is a quasi-instance of Q33; the quasi-statement:

$$(w)(\mathrm{F}(w,y) \supset \sim\mathrm{F}(y,w)) \supset (\mathrm{F}(y,y) \supset \sim\mathrm{F}(y,y)) \qquad (28),$$

needed in 2, is not. To reach (28), we must assume by R5 the following quasi-instance of Q33:

$$(x)(\mathrm{F}(x,y) \supset \sim\mathrm{F}(y,x)) \supset (\mathrm{F}(y,y) \supset \sim\mathrm{F}(y,y)) \qquad (29),$$

and deduce (28) as a relettering of (29) by R6.

R6 is easily justified. It can indeed be shown that if ψ is a relettering of a formula φ, then ψ is logically equivalent to φ; but if ψ is logically equivalent to φ, then ψ is deducible from φ; hence if ψ is a relettering of φ, then ψ is deducible from φ. Note, however, that if ψ is to count as a relettering of φ, then the variable which ψ shows in place of a bound variable of φ must, as stated above, be foreign to φ. Take, for instance,

$$(\mathrm{E}z)(z \neq x) \qquad (30).$$

The result of substituting 'x' for all the occurrences of 'z' bound in (30) by $(\mathrm{E}z)$', namely:

$$(\mathrm{E}x)(x \neq x) \qquad (31),$$

is false, whereas (30) itself may be true. (31) may accordingly not count as a relettering of (30).

There is still one gap in the foregoing proof. Instead of deducing from

$$(x)(w)(\mathrm{F}(w,x) \supset \sim\mathrm{F}(x,w)) \qquad (23)$$

the conclusion:

$$(y)\sim\mathrm{F}(y,y) \qquad (24),$$

we deduced the conclusion:

$$\sim\mathrm{F}(y,y) \qquad (25).$$

To pass from the quasi-statement (25) to the statement (24) we need a new rule, *the rule of Universalization*. It reads:

R7: "From φ one may deduce ψ, if ψ is the result of writing a universal quantifier in front of φ."

With R7 at hand, we may write '(y)' in front of (25) and thereby obtain (24), our final conclusion.

R7 enables us to turn a quasi-statement into a statement and close our deduction, as expected, with a statement. The closures of a quasi-

statement are not necessarily true; the two closures: '$(x)(y)(x = y \supset y \neq x)$' and '$(y)(x)(x = y \supset y \neq x)$', of '$x = y \supset y \neq x$', for example, are false. But the only quasi-statements which may enter a deduction

1. are instances of valid schemata, like '$(x)(w)(\mathrm{F}(w,x) \supset \sim\mathrm{F}(x,w)) \supset (w)(\mathrm{F}(w,y) \supset \sim\mathrm{F}(y,w))$';

or 2. follow from premises and instances of valid schemata, like '$(w)(\mathrm{F}(w),y) \supset \sim\mathrm{F}(y,w))$'.

In the first case their closures are logically true; in the second, they are true if, as assumed, the premises from which they follow are true. In either case, therefore, our use of R7 to turn quasi-statements into statements is justified.

Our seven rules of quantificational deduction may be summed up as follows:

R1: "From φ and $\ulcorner\varphi \supset \psi\urcorner$ one may deduce ψ (Detachment)."
R2: "From φ and ψ one may deduce $\ulcorner\varphi \,.\, \psi\urcorner$ (Adjunction)."
R3: "If φ is an instance or a quasi-instance of a recorded sentential schema, then φ may serve as a premise in any quantificational deduction (Sentential Insertion)."
R4: "From φ and $\ulcorner\psi \equiv \chi\urcorner$ one may deduce ω, if ω is the result of interchanging ψ and χ at one or more places in φ (Interchange)."
R5: "If φ is an instance or a quasi-instance of a recorded quantificational schema, then φ may serve as a premise in any quantificational deduction (Quantificational Insertion)."
R6: "From φ one may deduce ψ, if ψ is a relettering of φ (Relettering)."
R7: "From φ one may deduce ψ, if ψ is the result of writing a universal quantifier in front of φ (Universalization)."

It can be shown that (*a*) if a conclusion ψ is deduced from a set of premises $\varphi_1, \varphi_2, \ldots, \varphi_n$ in accordance with one or more of R1–R7, then

$$\ulcorner(\varphi_1 \,.\, \varphi_2 \,.\, \ldots \,.\, \varphi_n) \supset \psi\urcorner$$

is sententially or quantificationally true, and hence the set of premises $\varphi_1, \varphi_2, \ldots, \varphi_n$ logically implies ψ. It can be shown conversely that (*b*) if

$$\ulcorner(\varphi_1 \,.\, \varphi_2 \,.\, \ldots \,.\, \varphi_n) \supset \psi\urcorner$$

is sententially or quantificationally true and hence the set of premises $\varphi_1, \varphi_2, \ldots, \varphi_n$ logically implies ψ, then ψ can be deduced from $\varphi_1, \varphi_2, \ldots, \varphi_n$ in accordance with one or more of R1–R7. Our seven rules are thus adequate for all deductive purposes in sentential and quantificational logic.[10]

We are left with a last problem: to determine which formulae may

[10]As remarked in chapter one, R2 and R4 may be dispensed with.

count as an instance or a quasi-instance of T1–T24c and Q30a–Q39c. We shall study it in three paragraphs respectively entitled *sentential substitution, argument substitution,* and *predicate substitution.*

1. *Sentential substitution.* The result of substituting a statement for a sentential dummy in a formula φ always counts as an instance of φ; so does, among others, the result of substituting 'John is silly' for 'p' in T3b:

$$\text{John is silly} \equiv \text{John is silly}.$$

The result, however, of substituting a quasi-statement for a sentential dummy in a formula φ may not always count as an instance of φ. Take, for example, the recorded schema:

$$(\mathrm{E}x)(p \vee F(x)) \equiv (p \vee (\mathrm{E}x)F(x)).$$

The result of substituting '$y = y$' for 'p' (and, say, '$x \neq x$' for '$F(x)$') in Q37g, namely:

$$(\mathrm{E}x)(y = y \vee x \neq x) \equiv (y = y \vee (\mathrm{E}x)(x \neq x)),$$

may count as an instance of it. The result, however, of substituting '$x < 1$' for 'p' (and, again, '$x \neq x$' for '$F(x)$') in the same schema, namely:

$$(\mathrm{E}x)(x < 1 \vee x \neq x) \equiv (x < 1 \vee (\mathrm{E}x)(x \neq x)) \qquad (32),$$

may not count as an instance of it. The trouble with this substitution is that the free 'x' of '$x < 1$' is caught at its first occurrence in (32) by the opening quantifier '$(\mathrm{E}x)$' and turned into a bound variable. Such a mishap can be avoided by substituting for 'p' in Q37g only quasi-statements like '$y = y$' no free variable of which is already bound in Q37g. We may therefore state: "The result of substituting a statement or quasi-statement ψ for a sentential dummy in a formula φ may count as an instance of φ, if no free variable of ψ is already bound in φ."

2. *Argument substitution.* The result of substituting an argument for a free argument variable in a formula φ always counts as an instance of φ; so does, among others, the result of substituting 'Socrates' for 'y' (and, say, 'is a man' for 'F') in Q34:

$$\text{Socrates is a man} \supset (\mathrm{E}x)(x \text{ is a man}).$$

The result, however, of substituting an argument variable for a free argument variable in a formula φ may not always count as an instance of φ. Take the following instance of Q33:

$$(x)(\mathrm{E}z)(z \neq x) \supset (\mathrm{E}z)(z \neq y) \qquad (33).$$

The result of substituting 'w' for 'y' in (33), namely:

$$(x)(\mathrm{E}z)(z \neq x) \supset (\mathrm{E}z)(z \neq w),$$

may count as an instance of Q33. The result, however, of substituting 'z' for 'y' in (33), namely:

$$(x)(\mathrm{E}z)(z \neq x) \supset (\mathrm{E}z)(z \neq z),$$

may not count as an instance of Q33. The trouble with this substitution is that the 'z' substituted for the free 'y' is caught by the second occurrence of '(Ez)' and turned into a bound variable. Such a mishap can easily be avoided by substituting for 'y' in (33) only argument variables like 'w' which are not already bound in (33). We may therefore state:

"The result of substituting an argument for a free argument variable in a formula φ may count as an instance of φ. The result of substituting an argument variable for a free argument variable in a formula φ may count as an instance of φ, if the former argument variable is not already bound in φ."

3. *Predicate substitution.* Predicate substitution was defined above as the substitution of a predicate for a predicate dummy in a quantificational schema. For convenience's sake we shall modify our terminology here and treat predicate substitution as the substitution of statements or quasi-statements for the atomic components of a quantificational schema.

"(a) Let a quantificational schema φ and an n-adic ($n \geq 1$) predicate dummy be given, and let $\chi_1, \chi_2, \ldots, \chi_j$ ($j \leq 1$) be, in the order of their appearance in φ, all the atomic components of φ which open with the predicate dummy in question. (b) Let a statement or a quasi-statement ψ and n distinct argument variables, to be called *the substitution variables of* ψ, be given and let all the argument variables of ψ other than its substitution variables be called *the non-substitution variables of* ψ. (c) Let ψ_i, for each i from 1 to j, be the result of replacing every free occurrence in ψ of the first substitution variable of ψ by the first argument variable occurring in χ_i, every free occurrence of the second substitution variable of ψ by the second argument variable occurring in $\chi_i, \ldots$, and every free occurrence of the n-th substitution variable of ψ by the n-th argument variable occurring in χ_i. (d) Finally, let φ' be the result of replacing χ_1 in φ by ψ_1, χ_2 by ψ_2, $\ldots$, and χ_n by ψ_n. (e) If none of the argument variables of φ is already bound in ψ nor any of the free non-substitution variables of ψ is already bound in φ, then φ' may count as an instance of φ."

Let us first illustrate clauses (a)–(d) of the above statement.

Example 1. Let φ be Q35a: '$(x)F(x) \supset (\mathrm{E}x)F(x)$'; both χ_1 and χ_2 will be '$F(x)$'.

1a. Let ψ be 'x is self-identical' and let its substitution variable be 'x'; then ψ_1 and ψ_2 will both be 'x is self-identical', and φ' will be '(x)(x is self-identical) $\supset$ (Ex)(x is self-identical)'.

1*b*. Let ψ be '$x = y$' and let its substitution variable be 'x'; then ψ_1 and ψ_2 will both be '$x = y$', and φ' will be '$(x)(x = y) \supset (\mathrm{E}x)(x = y)$'.

1*c*. Let ψ be '$x = y$' and let its substitution variable be 'y'; then ψ_1 and ψ_2 will both be '$x = x$', and φ' will be '$(x)(x = x) \supset (\mathrm{E}x)(x = x)$'.

Example 2. Let φ be Q33: '$(x)F(x) \supset F(y)$'; χ_1 will be '$F(x)$' and χ_2 will be '$F(y)$'.

2*a*. Let ψ be 'x is self-identical' and let its substitution variable be 'x'; then ψ_1 will be 'x is self-identical', ψ_2 will be 'y is self-identical', and φ' will be '$(x)(x$ is self-identical$) \supset y$ is self-identical'.

2*b*. Let ψ be '$x = y$' and let its substitution variable be 'x'; then ψ_1 will be '$x = y$', ψ_2 will be '$y = y$', and φ' will be '$(x)(x = y) \supset y = y$'.

2*c*. Let ψ be '$x = y$' and let its substitution variable be 'y'; then ψ_1 will be '$x = x$', ψ_2 will be '$x = y$', and φ' will be '$(x)(x = x) \supset x = y$'.

2*d*. Let ψ be '$w = z$' and let its substitution variable be 'w'; then ψ_1 will be '$x = z$', ψ_2 will be '$y = z$', and φ' will be '$(x)(x = z) \supset y = z$'.

Example 3. Let φ be Q39a: '$(x)(y)F(x,y) \equiv (y)(x)F(x,y)$'; both χ_1 and χ_2 will be '$F(x,y)$'.

3*a*. Let ψ be '$x = y$' and let its first and second substitution variables respectively be 'x' and 'y'; then ψ_1 and ψ_2 will both be '$x = y$', and φ' will be '$(x)(y)(x = y) \equiv (y)(x)(x = y)$'.

3*b*. Let ψ be '$x = y$' and let its first and second substitution variables respectively be 'y' and 'x'; then ψ_1 and ψ_2 will both be '$y = x$', and φ' will be '$(x)(y)(y = x) \equiv (y)(x)(y = x)$'.

3*c*. Let ψ be '$w = w' \times z$' and let its first and second substitution variables respectively be 'w' and 'z'; then ψ_1 and ψ_2 will both be '$x = w' \times y$', and φ' will be '$(x)(y)(x = w' \times y) \equiv (y)(x)(x = w' \times y)$'.

In the above examples the substitution variables of ψ were all chosen among the free variables of ψ.[11] Note, however, that some or all of the substitution variables of ψ may be chosen outside the free variables of ψ and that ψ itself need not contain n free variables. Let φ, for instance, be Q39a: '$(x)(y)F(x,y) \equiv (y)(x)F(x,y)$'. (*a*) If ψ is '$x = y$' and its substitution variables are any two argument variables but 'x' and 'y', then ψ_1 and ψ_2 are ψ, and φ' is '$(x)(y)(x = y) \equiv (y)(x)(x = y)$'; (*b*) if ψ is '$w = z$' and its substitution variables are any two argument variables but 'w' and 'z', then ψ_1 and ψ_2 are ψ, and φ' is '$(x)(y)(w = z) \equiv (y)(x)(w = z)$'; (*c*) if, finally, ψ is 'John is sick' and its substitution variables are any two argument variables, then ψ_1 and ψ_2 are ψ, and φ' is '$(x)(y)$(John is sick) $\equiv (y)(x)$(John is sick)'. Observe also that if the substitution variables

[11]Similarly, in the quantificational deductions offered below the substitution variables of ψ will all be chosen among the free variables of ψ.

of ψ are chosen outside the free variables of ψ and ψ is m-adic ($m < n$), then each χ_i ($i = 1, 2, \ldots, j$) will have more argument-places than ψ_i does; in substitution (c), for example, each χ_i has two argument-places, whereas ψ_i has only one argument-place. Hence the possibility, already noted above, of an atomic quantificational schema having more argument-places than some of its instances do.

Let us now turn to the two if-clauses of (e).

1. Let φ be Q33: '$(x)F(x) \supset F(y)$'; let ψ be '$(\mathrm{E}y)(y \neq x)$' and let its substitution variable be 'x'. Then ψ_1 will be '$(\mathrm{E}y)(y \neq x)$', ψ_2 will be '$(\mathrm{E}y)(y \neq y)$', and φ' will be '$(x)(\mathrm{E}y)(y \neq x) \supset (\mathrm{E}y)(y \neq y)$', a conditional whose antecedent may be true, but whose consequent is false; φ' may accordingly not count as an instance of φ. Note here that one of the variables of φ, namely 'y', is bound in ψ.
2. Let φ be Q34: '$F(y) \supset (\mathrm{E}x)F(x)$'; let ψ be '$w \neq x$' and let its substitution variable be 'w'. Then ψ_1 will be '$y \neq x$', ψ_2 will be '$x \neq x$', and φ' will be '$y \neq x \supset (\mathrm{E}x)(x \neq x)$', a conditional whose antecedent may be true but whose consequent is false; φ' may accordingly not count as an instance of φ. Note here that one of the free non-substitution variables of ψ, namely 'x', is bound in φ.

Predicate substitutions, though delicate to perform in other contexts, are easily performed here. Observe indeed that Q30a–Q37l contain only monadic predicate dummies and Q38a–Q39c only dyadic predicate dummies; the former schemata accordingly call for only one substitution variable *per* predicate dummy, the latter for only two substitution variables *per* predicate dummy. Note also that in any recorded schema at most two atomic components χ_i open with the same predicate dummy, be it 'F', 'G', or 'H', and that in any recorded schema but Q33 and Q34 the two χ_i's in question, if two there are, are identical; hence in any recorded schema at most two predicate substitutions *per* predicate dummy need be performed and in any recorded schema but Q33 and Q34 the two substitutions in question, if two need be performed, are identical.

19. SAMPLE DEDUCTIONS

A *quantificational deduction* is a series of formulae, called *steps*, each member of which:

1. is a statement assumed as a premise,[12]

[12]We might enlarge the present concept of a quantificational deduction by allowing both statements and quasi-statements to function as premises in 1. If we did so, however, we should modify R7 to read: "From φ one may deduce ψ, if ψ is the result of writing a universally quantified variable in front of φ and the variable in question is not free in any premise (Universalization)." For an explanation of this technicality, cf. chapter three, section 29.

2. is an instance or a quasi-instance of a recorded schema,

or 3. follows from previous members of the series through R1, R2, R4, R6, or R7;

the last member of a quantificational deduction is what we called above a *conclusion.*

We adopt the following conventions for recording quantificational deductions:

1. Each formula entering a deduction will be written on a separate line and numbered, except for the conclusion which is preceded by the two letters 'Cl'; premises and formulae assumed through R3 or R5 will also be preceded by the letter 'P'.

2. After each step of a deduction will be quoted both the formulae from which and the rule of deduction through which that step follows. R1, R2, and R4 may be quoted as in chapter one.

(*a*) Relettering (R6) will be quoted as in the following sample:

n: $(x)(x = x) \supset (x)(x = x)$	...
$n + 1$: $(y)(y = y) \supset (x)(x = x)$	Rel of n; y/x.

(*b*) Universalization (R7) will be quoted as in the following sample:

n: $x = x$	...
$n + 1$: $(x)(x = x)$	Univ of n.

3. The substitutions performed in a recorded schema φ to obtain an instance or a quasi-instance φ' of φ will be quoted to the right of φ' along with the letters 'Ins' and the reference number of φ.

(*a*) Sentential substitutions may be quoted as in the following sample:

Pn: $(x)(x = x) \supset (x)(x = x)$	Ins T3a; $(x)(x = x)/p$.

(*b*) Argument substitutions may be quoted as in the following sample:

Pn: $(x)(x = x) \supset z = z$	Ins Q33; z/y, . . . ,

where the blank '. . .' is to be filled by a predicate substutition as in (*c*).

(*c*) Predicate substitutions may be quoted after the following pattern:

Pn: φ'	Ins φ; ψ/χ_1,

where φ, χ_1, ψ, and φ' are as on page 86.

Sample substitutions:

Pn: $(x)(x = x) \supset y = y$	Ins Q33; $x = x/F(x)$,
Pn: $y = y \supset (\text{E}x)(x = x)$	Ins Q34; $x = x/F(y)$.

Conventions on substitution variables: To simplify matters, we shall

assume here that all the substitution variables of ψ are free variables of ψ.
Case 1: χ_1 is a monadic schema.

1*a*. If ψ has only one free variable, we shall automatically use that variable as the substitution variable of ψ.

1*b*. If ψ has n ($n > 1$) free variables, we shall underline once the free variable of ψ which is to serve as its substitution variable. Sample substitution:

Pn: $(y \leq z \,.\, y \neq z) \supset (\mathrm{E}x)(x \leq z \,.\, x \neq z)$ — Ins Q34; $\underline{w} \leq z \,.\, \underline{w} \neq z/F(y)$.

Case 2: χ_1 is a dyadic schema. When χ_1 is a dyadic schema, we shall underline once the free variable of ψ which is to serve as its first substitution variable and underline twice the free variable of ψ which is to serve as its second substitution variable. Sample substitution:

Pn: $(x)(y)(y \leq x \,.\, x \neq z) \equiv (y)(x)(y \leq x \,.\, x \neq z)$ — Ins Q39a; $\underline{\underline{w}} \leq \underline{x} \,.\, \underline{x} \neq z/F(x,y)$.

4. To condense deductions, we shall use the reference numeral of a given step, say 'n', as an abbreviation for step n; when step n is a conditional or a biconditional, we shall also use 'Ln' as an abbreviation for its left-hand component and 'Rn' as an abbreviation for its right-hand component. In example 1, below, for instance,

$$(1 \,.\, 2) \supset (x)(\mathrm{A}(x) \supset \mathrm{N}(x))$$

in short for

$$((x)(\mathrm{M}(x) \supset \mathrm{N}(x)) \,.\, (x)(\mathrm{A}(x) \supset \mathrm{M}(x))) \supset (x)(\mathrm{A}(x) \supset \mathrm{N}(x)),$$

and

$$\mathrm{L}3$$

is short for

$$(x)(\mathrm{M}(x) \,.\, \mathrm{N}(x)) \,.\, (x)(\mathrm{A}(x) \supset \mathrm{M}(x)).$$

To illustrate the above conventions, we shall set up six quantificational deductions; the first three belong to monadic quantificational logic, the last three to polyadic quantificational logic.

Example 1: As our first example we shall complete the deduction outlined on page 79.

P1: $(x)(\mathrm{M}(x) \supset \mathrm{N}(x))$	
P2: $(x)(\mathrm{A}(x) \supset \mathrm{M}(x))$	
P3: $(1 \,.\, 2) \supset (x)(\mathrm{A}(x) \supset \mathrm{N}(x))$	Ins Q32a; $\mathrm{M}(x)/G(x)$, $\mathrm{N}(x)/H(x)$, $\mathrm{A}(x)/F(x)$
4: L3	Adj 1 and 2
Cl: $(x)(\mathrm{A}(x) \supset \mathrm{N}(x))$	Det 3 and 4

This deduction, as the reader may check, calls only for R1, R2, and R5. So does the following one.

Example 2: As our second example, we shall formalize the deduction sketched on page 80. We are given the three premises:

P1: $(x)(x$ is a man $\supset x$ is mortal$)$,
P2: $(x)(x$ is an American $\supset x$ is a man$)$,
P3: $(\mathrm{E}x)(x$ is an American$)$,

and asked to deduce from them the conclusion:

Cl: $(\mathrm{E}x)(x$ is an American $. \ x$ is mortal$)$.

We use the same abbreviations as in example 1.

P1:	$(x)(\mathrm{M}(x) \supset \mathrm{N}(x))$	
P2:	$(x)(\mathrm{A}(x) \supset \mathrm{M}(x))$	
P3:	$(\mathrm{E}x)\mathrm{A}(x)$	
P4:	$(1 \,.\, 2) \supset (x)(\mathrm{A}(x) \supset \mathrm{N}(x))$	Ins Q32a; $\mathrm{M}(x)/G(x)$, $\mathrm{N}(x)/H(x)$, $\mathrm{A}(x)/F(x)$
P5:	$(\mathrm{R}4 \,.\, 3) \supset (\mathrm{E}x)(\mathrm{A}(x) \,.\, \mathrm{N}(x))$	Ins Q35b; $\mathrm{A}(x)/F(x)$, $\mathrm{N}(x)/G(x)$
6:	L4	Adj 1 and 2
7:	R4	Det 4 and 6
8:	L5	Adj 7 and 3
Cl:	$(\mathrm{E}x)(\mathrm{A}(x) \,.\, \mathrm{N}(x))$	Det 8 and 5

Example 3: As our third example, we shall deduce the conclusion:

Cl: If Jones majors in Philosophy and does not minor in Psychology, then he must take a course in Classical Archaeology,

from the two premises:

P1: All those who major in Philosophy must minor in Psychology or in Greek,
P2: All those who minor in Greek or in Latin must take a course in Classical Archaeology.

We abbreviate 'x majors in Philosophy' as '$\mathrm{P}(x)$', 'x minors in Psychology' as '$\mathrm{S}(x)$', 'x minors in Greek' as '$\mathrm{K}(x)$', 'x minors in Latin' as '$\mathrm{L}(x)$', 'x takes a course in Classical Archaeology' as '$\mathrm{C}(x)$', and 'Jones' as 'j'.

P1:	$(x)(\mathrm{P}(x) \supset (\mathrm{S}(x) \vee \mathrm{K}(x)))$	
P2:	$(x)((\mathrm{K}(x) \vee \mathrm{L}(x)) \supset \mathrm{C}(x))$	
P3:	$\mathrm{S}(x) \equiv \sim\sim \mathrm{S}(x)$	Ins. T5; $\mathrm{S}(x)/p$
4:	$(x)(\mathrm{P}(x) \supset (\mathrm{R}3 \vee \mathrm{K}(x)))$	Int 1 and 3

P5:	$(\sim \mathrm{S}(x) \supset \mathrm{K}(x)) \equiv (\mathrm{R3} \vee \mathrm{K}(x))$	Ins T18; $\sim \mathrm{S}(x)/p$, $\mathrm{K}(x)/q$
6:	$(x)(\mathrm{P}(x) \supset \mathrm{L5})$	Int 4 and 5
P7:	$((\mathrm{P}(x) \,.\, \sim \mathrm{S}(x)) \supset \mathrm{K}(x) \equiv (\mathrm{P}(x) \supset \mathrm{L5})$	Ins T13a; $\mathrm{P}(x)/p$, $\sim \mathrm{S}(x)/q$, $\mathrm{K}(x)/r$
8:	$(x)\mathrm{L7}$	Int 6 and 7
P9:	$((\mathrm{K}(x) \vee \mathrm{L}(x)) \supset \mathrm{C}(x)) \equiv ((\mathrm{K}(x) \supset \mathrm{C}(x)) \,.\, (\mathrm{L}(x) \supset \mathrm{C}(x)))$	Ins T10e; $\mathrm{K}(x)/p$, $\mathrm{L}(x)/q$, $\mathrm{C}(x)/r$
10:	$(x)\mathrm{R9}$	Int 2 and 9
P11:	$10 \equiv ((x)(\mathrm{K}(x) \supset \mathrm{C}(x)) \,.\, (x)(\mathrm{L}(x) \supset \mathrm{C}(x)))$	Ins Q36a; $\mathrm{K}(x) \supset \mathrm{C}(x)/F(x)$, $\mathrm{L}(x) \supset \mathrm{C}(x)/G(x)$
12:	R11	Int 10 and 11
P13:	$12 \supset (x)(\mathrm{K}(x) \supset \mathrm{C}(x))$	Ins T6a; $(x)(\mathrm{K}(x) \supset \mathrm{C}(x))/p$, $(x)(\mathrm{L}(x) \supset \mathrm{C}(x))/q$
14:	R13	Det 12 and 13
P15:	$(14 \,.\, 8) \supset (x)((\mathrm{P}(x) \,.\, \sim \mathrm{S}(x)) \supset \mathrm{C}(x))$	Ins Q32a; $\mathrm{K}(x)/G(x)$, $\sim \mathrm{S}(x)/F(x)$ and $\mathrm{C}(x)/H(x)$
16:	$14 \,.\, 8$	Adj 14 and 8
17:	R15	Det 15 and 16
P18:	$17 \supset ((\mathrm{P(j)} \,.\, \sim \mathrm{S(j)}) \supset \mathrm{C(j)})$	Ins Q33; $(\mathrm{P}(x) \,.\, \sim \mathrm{S}(x)) \supset \mathrm{C}(x)/F(x)$, j/y
Cl:	$(\mathrm{P(j)} \,.\, \sim \mathrm{S(j)}) \supset \mathrm{C(j)}$	Det 17 and 18

This deduction, as the reader may check, calls for R1 (14, 17, Cl), R2 (16), R3 (3, 5, 7, 9, 1), R4 (4, 6, 8, 10, 12), and R5 (11, 15, 18).

Example 4: As our fourth example, we shall formalize the deduction sketched on pages 82–83. The premise and conclusion read, we may remember:

P1: No one is the father of his own father;
Cl: No one is his own father.

Using the abbreviations already agreed on, we obtain:

P1:	$(x)(w)(\mathrm{F}(w,x) \supset \sim \mathrm{F}(x,w))$	
P2:	$1 \supset (w)(\mathrm{F}(w,y) \supset \sim \mathrm{F}(y,w))$	Ins Q33; $(w)(\mathrm{F}(w,x) \supset \sim \mathrm{F}(x,w))/F(x)$
3:	R2	Det 1 and 2
4:	$(x)(\mathrm{F}(x,y) \supset \sim \mathrm{F}(y,x))$	Rel of 3; x/w
P5:	$4 \supset (\mathrm{F}(y,y) \supset \sim \mathrm{F}(y,y))$	Ins Q33; $\mathrm{F}(x,y) \supset \sim \mathrm{F}(y,x)/F(x)$

6: R5	Det 4 and 5
P7: $6 \equiv (\sim F(y,y) \vee \sim F(y,y))$	Ins T18; $F(y,y)/p$, $\sim F(y,y)/q$
P8: $R7 \equiv \sim F(y,y)$	Ins T4b; $\sim F(y,y)/p$
9: $6 \equiv R8$	Int 7 and 8
10: R9	Int 6 and 9
Cl: $(y)\sim F(y,y)$	Univ of 10

Conspicuous in this deduction, as in the following ones, is the use of R6 and R7.

Example 5: As our fifth example, we shall deduce the conclusion:

Cl: Tails of horses are tails of animals,

from the single premise:

P1: Horses are animals.

Using '$H(x)$', '$A(x)$', and '$T(y,x)$' as abbreviations for 'x is a horse', 'x is an animal', and 'y is the tail of x', we shall respectively translate our premise and conclusion into:

P1: $(x)(H(x) \supset A(x))$;
Cl: $(y)((Ex)(T(y,x) \,.\, H(x)) \supset (Ex)(T(y,x) \,.\, A(x)))$.

The deduction may run as follows:

P1: $(x)(H(x) \supset A(x))$	
P2: $((x)(H(x) \supset A(x)) \,.\, (Ex)(T(y,x) \,.\, H(x))) \supset (Ex)(T(y,x) \,.\, A(x))$	Ins Q32b; $H(x)/G(x)$, $A(x)/H(x)$, $T(y,x)/F(x)$
P3: $2 \equiv ((x)(H(x) \supset A(x)) \supset ((Ex)(T(y,x) \,.\, A(x)) \supset R2)$	Ins T13a; $(x)(H(x) \supset A(x))/p$, $(Ex)(T(y,x) \,.\, H(x))/q$, $R2/r$
4: R3	Int 2 and 3
5: R4	Det 1 and 4
Cl: $(y)((Ex)(T(y,x) \,.\, H(x)) \supset (Ex)(T(y,x) \,.\, A(x)))$	Univ of 5

Example 6: As our last example, we shall deduce the conclusion:

Cl: No one dislikes anyone who likes him and no one likes anyone who dislikes him,

from the two premises:

P1: One is prejudiced against anyone liked by someone he dislikes,
P2: No one is prejudiced against one's self.

Using '$L(x,y)$', '$\sim L(x,y)$', and '$P(x,y)$' as abbreviations for 'x likes y', 'x dislikes y', and 'x is prejudiced against y', we shall respectively translate our premises and conclusion into:

P1: $(x)(y)(z)((\sim L(x,y) \,.\, L(y,z)) \supset P(x,z),$
P2: $(x){\sim}P(x,x),$
C1: $(x)(y)(L(y,x) \supset L(x,y)) \,.\, (x)(y)(\sim L(y,x) \supset \sim L(x,y))$

The deduction may run as follows:

P1:	$(x)(y)(z)((\sim L(x,y) \,.$ $L(y,z)) \supset P(x,z))$	
P2:	$(x){\sim}P(x,x)$	
P3:	$1 \supset (y)(z)((\sim L(w,y) \,.$ $L(y,z)) \supset P(w,z))$	Ins Q33; $(y)(z)((\sim L(x,y) \,.$ $L(y,z)) \supset P(x,z)/F(x)$
4:	R3	Det 1 and 3
5:	$(x)(z)((\sim L(w,x) \,.$ $L(x,z)) \supset P(w,z))$	Rel of 4; x/y
P6:	$5 \supset (z)((\sim L(w,y) \,.$ $L(y,z)) \supset P(w,z))$	Ins Q33; $(z)((\sim L(w,\underline{x}) \,.$ $L(x,z)) \supset P(w,z))/F(x)$
7:	R6	Det 5 and 6
8:	$(x)((\sim L(w,y) \,.\, L(y,x)) \supset P(w,x))$	Rel of 7; x/z
P9:	$8 \supset ((\sim L(w,y) \,.$ $L(y,w)) \supset P(w,w))$	Ins Q33; $(\sim L(w,y) \,.$ $L(y,\underline{x})) \supset P(w,x)/F(x)$
10:	R9	Det 8 and 9
P11:	$2 \supset \sim P(w,w)$	Ins Q33; $\sim P(x,x)/F(x)$, w/y
12:	R11	Det 2 and 11
13:	$10 \,.\, 12$	Adj 10 and 12
P14:	$13 \supset \sim(\sim L(w,y) \,.\, L(y,w))$	Ins T24a; $\sim L(w,y) \,.\, L(y,w)/p$, $P(w,w)/q$
15:	R14	Det 13 and 14
P16:	$(\sim L(w,y) \,.\, L(y,w)) \equiv$ $(L(y,w) \,.\, \sim L(w,y))$	Ins T8a; $\sim L(w,y)/p$, $L(y,w)/q$
17:	$\sim$R16	Int 15 and 16
P18:	$(L(y,w) \supset L(w,y)) \equiv 17$	Ins T19; $L(y,w)/p$, $L(w,y)/q$
19:	L18	Int 17 and 18
20:	$(y)(L(y,w) \supset L(w,y))$	Univ of 19
21:	$(w)(y)(L(y,w) \supset L(w,y))$	Univ of 20
22:	$(x)(y)(L(y,x) \supset L(x,y))$	Rel of 21; x/y
P23:	$(L(y,w) \supset L(w,y)) \equiv$ $(\sim L(w,y) \supset \sim L(y,w))$	Ins T16a; $L(y,w)/p$, $L(w,y)/q$
24:	$(w)(y)(\sim L(w,y) \supset \sim L(y,w))$	Int 21 and 23

25:	$(w)(x)(\sim L(w,x) \supset \sim L(x,w))$	Rel of 24; x/y
26:	$(y)(x)(\sim L(y,x) \supset \sim L(x,y))$	Rel of 25; y/w
P27:	$(x)(y)(\sim L(y,x) \supset \sim L(x,y)) \equiv 26$	Ins Q39a; $\sim L(\underline{y},\underline{\underline{x}}) \supset \sim L(x,y)/F(x,y)$
28:	L27	Int 26 and 27
Cl:	$(x)(y)(L(y,x) \supset L(x,y))\ .\ (x)(y)(\sim L(y,x) \supset \sim L(x,y))$	Adj 22 and 28

*20. QUANTIFICATIONAL VALIDITY

In section 17 we called a quantificational schema *valid* when its instances were true. This definition must now be refined in one respect.

The quantified statement:

$(x)(x$ is an even number $\supset x$ is divisible by 2),

is true whatever be the values of the argument variable 'x'; the quantified statement:

$(x)(x$ is divisible by 2),

on the other hand, is true if 'x' is understood to take even numbers as its values, false otherwise. Similarly, the quantified statement:

$(x)(x$ is a man $\supset x$ is a featherless biped),

is true whatever be the values of the argument variable 'x'; the quantified statement:

$(x)(x$ is a featherless biped),

on the other hand, is true if 'x' is understood to take men as its values, false otherwise. Some quantified statements may thus be true when their argument variables take certain entities as their values, false when they take others.

Let us call any non-empty set of entities a *universe of discourse* and let us call a statement φ *true of a given universe of discourse* D if φ is true when its argument variables take the members of D as their values.[13] We shall say that a quantificational schema is *valid* if its instances are true of all universes of discourse.

Universes of discourse may be finite or infinite in size; the universe of discourse consisting of all divisors of 24, for example, is finite in size, the universe of discourse consisting of all even numbers infinite in size. We shall say that a quantificational schema is *n-valid* ($n = 1, 2, 3, \ldots$)

[13]The predicate 'is true of' here defined should not be confused with the predicate 'is true of' defined in section 17; the former takes a predicate as its subject, the latter a statement.

if its instances are true of all universes of discourse of size n, and *f-valid* if its instances are true of all finite universes of discourse.

There is a mechanical way of checking whether a schema φ is n-valid or not for any given n. It consists of the following steps:

1. Write in front of φ universal quantifiers binding all the free argument variables of φ; let the resulting schema be φ_1.

2. Replace every universally quantified component, say '$(x)($—— x ——$)$', of φ_1 by:

$$(\ldots(\text{——}\ a_1\ \text{——}\ .\ \text{——}\ a_2\ \text{——})\ .\ \ldots)\ .\ \text{——}\ a_n\ \text{——},$$

where 'a_1', 'a_2', ... , 'a_n' are n arbitrary arguments; let the resulting schema be φ_2.

3. Replace every existentially quantified component, say '$(\mathrm{E}x)$ $($——x——$)$', of φ_2 by:

$$(\ldots(\text{——}\ a_1\ \text{——}\ \mathsf{v}\ \text{——}\ a_2\ \text{——})\ \mathsf{v}\ \ldots)\ .\ \text{——}\ a_n\ \text{——},$$

where 'a_1', 'a_2', ... , 'a_n' are the n arguments listed in 2; let the resulting schema be φ_3.

4. Subject φ_3 to the truth-table test.

It is easily seen that φ is n-valid if and only if φ_3 is a tautology.

Examples:

(*a*) Let φ be:

$$F(y) \supset (\mathrm{E}x)F(x),$$

and let n be 2.

1. φ_1 is '$(y)(F(y) \supset (\mathrm{E}x)F(x))$';
2. φ_2 is '$(F(a_1) \supset (\mathrm{E}x)F(x))\ .\ (F(a_2) \supset (\mathrm{E}x)F(x))$';
3. φ_3 is '$(F(a_1) \supset (F(a_1) \mathsf{v} F(a_2)))\ .\ (F(a_2) \supset (F(a_1) \mathsf{v} F(a_2))$';
4. φ_3 is a tautology, as the reader may check; φ is therefore 2-valid.

(*b*) Let φ be:

$$(\mathrm{E}x)(y)F(x,y) \supset (y)(\mathrm{E}x)F(x,y),$$

and let n be 2.

1. φ_1 is '$(\mathrm{E}x)(y)F(x,y) \supset (y)(\mathrm{E}x)F(x,y)$';
2. φ_2 is '$(\mathrm{E}x)(F(x,a_1)\ .\ F(x,a_2)) \supset ((\mathrm{E}x)F(x,a_1)\ .\ (\mathrm{E}x)F(x,a_2))$';
3. φ_3 is '$((F(a_1,a_1)\ .\ F(a_1,a_2))\ \mathsf{v}\ (F(a_2,a_1)\ .\ F(a_2,a_2))) \supset ((F(a_1,a_1)$ $\mathsf{v}\ F(a_2,a_1))\ .\ (F(a_1,a_2)\ \mathsf{v}\ F(a_2,a_2)))$';
4. φ_3 is a tautology, as the reader may check; φ is therefore 2-valid.

It is clear that a quantificational schema may be n-valid for a certain n without being n-valid for a larger n; the schema:

$$(\mathrm{E}x)F(x) \supset (x)F(x),$$

for instance, though 1-valid, is not 2-valid, as the reader may check.

Similarly, a quantificational schema may be f-valid without being valid; the schema:

$$\sim((x)\sim F(x,x) \,.\, ((x)(y)(z)((F(x,y) \,.\, F(y,z)) \supset F(x,z)) \,.\, (x)(\mathrm{E}y)F(x,y))),$$

for instance, though f-valid, is not valid. It can be proved, however, that if a monadic schema φ is 2^k-valid, where k is the number of distinct predicate dummies occurring in φ, then φ is valid. The above test may accordingly be used as a validity test for monadic schemata.[14]

Example 1: Let φ be the monadic schema:

$$(x)F(x) \equiv \sim(\mathrm{E}x)\sim F(x) \qquad (1).$$

(1) has one predicate dummy, 'F'; to be valid it must therefore be 2^1-valid. φ_2 is '$(F(a_1) \,.\, F(a_2)) \equiv \sim(\mathrm{E}x)\sim F(x)$' and φ_3 is '$(F(a_1) \,.\, F(a_2)) \equiv \sim(\sim F(a_1) \vee \sim F(a_2))$', which is shown to be a tautology by the following truth-table:

$((a_1)$	$F(a_2)$	$(F(a_1) \,.\, F(a_2)) \equiv \sim(\sim F(a_1) \vee \sim F(a_2))$					
T	T	T	T	T	F	F	F
F	T	F	T	F	T	T	F
T	F	F	T	F	F	T	T
F	F	F	T	F	T	T	T

(1) is therefore valid.

Example 2: Let φ be the monadic schema:

$$(x)(F(x) \supset p) \equiv ((\mathrm{E}x)F(x) \supset p) \qquad (2).$$

(2) has one predicate dummy, 'F'; to be valid it must therefore be 2^1-valid. φ_2 is '$((F(a_1) \supset p) \,.\, (F(a_2) \supset p)) \equiv ((\mathrm{E}x)F(x) \supset p)$', and φ_3 is '$((F(a_1) \supset p) \,.\, (F(a_2) \supset p)) \equiv ((F(a_1) \vee F(a_2)) \supset p)$', which is shown to be a tautology by the following truth-table:

$F(a\beta)$	$F(a_2)$	p	$((F(a_1) \supset p) \,.\, (F(a_2) \supset p)) \equiv ((F(a_1) \vee F(a_2) \supset p))$					
T	T	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	T	T	T	T	T	T	T
F	F	T	T	T	T	T	F	T
T	T	F	F	F	F	T	T	F
F	T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	T	F
F	F	F	T	T	T	T	F	T

(2) is therefore valid.

[14] A proof that a monadic schema, if 2^k-valid, is valid, will be given in chapter five, section 48.

Example 3: Let φ be the monadic schema:

$$(x)(F(x) \supset G(x)) \supset ((x)(Fx) \supset (x)G(x)) \qquad (3).$$

(3) has two predicate dummies, 'F' and 'G'; to be valid it must therefore be 2^2-valid. φ_2 is '$(((F(a_1) \supset G(a_1)) \,.\, (F(a_2) \supset G(a_2))) \,.\, ((F(a_3) \supset G(a_3)) \,.\, (F(a_4) \supset G(a_4)))) \supset (((F(a_1) \,.\, F(a_2)) \,.\, (F(a_3) \,.\, F(a_4))) \supset ((G(a_1) \,.\, G(a_2)) \,.\, (G(a_3) \,.\, G(a_4))))$', which may be shown to be a tautology as follows:

(a) If φ_2 is to be assigned an 'F', its antecedent must be assigned a 'T' while its consequent is assigned an 'F';

(b) If the consequent of φ_2 is to be assigned an 'F', then '$(F(a_1) \,.\, F(a_2)) \,.\, (F(a_3) \,.\, F(a_4))$' must be assigned a 'T' while '$(G(a_1) \,.\, G(a_2)) \,.\, (G(a_3) \,.\, G(a_4))$' is assigned an 'F';

(c) If '$(F(a_1) \,.\, F(a_2)) \,.\, (F(a_3) \,.\, F(a_4))$' is to be assigned a 'T', then each '$F(a_i)$' must be assigned a 'T';

(d) If '$(G(a_1) \,.\, (G(a_2)) \,.\, (G(a_3) \,.\, G(a_4))$' is to be assigned an 'F', then at least one '$G(a_i)$' must be assigned an 'F';

(e) If the antecedent of φ_2 is to be assigned a 'T', then each '$F(a_i) \supset G(a_i)$' must be assigned a 'T';

(f) If each '$F(a_i) \supset G(a_i)$' is to be assigned a 'T' and if, as shown in (c), each '$F(a_i)$' is assigned a 'T', then each '$G(a_i)$' must be assigned a 'T' in contradiction with (d).

φ_2 is therefore a tautology and (3) is valid.

There exist other validity tests for monadic schemata besides the above. There also exist validity tests for certain types of polyadic schemata; there does not exist, however, any validity test for all polyadic schemata, and the American logician A. Church showed in 1936 that there cannot exist such a test.

*21. INTUITIONIST LOGIC

Two-valued sentential logic is based upon the following three principles:

P1a: A statement has at least one of the two truth-values 'T' and 'F';
P1b: A statement has at most one of the two truth-values 'T' and 'F';
P2: The truth-values of a sentential compound are determined by the truth-values of its components.

Many-valued sentential logics retain P2, but enlarge P1a and P1b to read, respectively:

P1a′: A statement has at least one among a finite or infinite number of truth-values;

P1b′: A statement has at most one among a finite or infinite number of truth-values.

Modal sentential logics retain P1a and P1b, but renounce P2.

The logic we are now about to study, *intuitionist logic*,[15] retains P1b, but, oddly enough, renounces P1a and P2. Renouncing P1a, it denies some statements either one of the two truth-values 'T' and 'F', but instead of assigning them a third truth-value of their own, as three-valued logic does, it simply leaves them without any. Renouncing P1a, it must also renounce P2, for if any sentential compound has at least one truth-valueless component, then its truth-values can clearly not be determined by the truth-values of its components.

Intuitionist logic is led to renounce P1a because of its peculiar understanding of the quantifier '(Ex)', where the variable 'x' ranges over mathematical entities and, especially, numbers. To better appreciate this point, we shall study three illustrations.

1. Let us first consider the statement:

There exist 100 prime numbers of the form $2^n + 1$ ($n = 1, 2, 3, \ldots$) (1).

(1) has never been proved nor disproved. According to classical logic, (1) is still either true or false, even though neither known to be true nor known to be false, and the 100 prime numbers in question either exist or do not exist, even though they are neither known to exist nor known not to exist. According to intuitionist logic, (1) cannot be said to be either true or false until known to be either one or the other, and the 100 prime numbers in question cannot be said either to exist or not to exist until known either to do so or not to do so.

We gather from this first example that an instance of '(Ex)$F(x)$' is true in intuitionist logic if and only if it is known to be true. But there are various ways in which an instance of '(Ex)$F(x)$', say '(Ex)(—— x ——)', can come to be known to be true. Intuitionist logic admits only such demonstrations of '(Ex)(—— x ——)' as provide means of calculating the x of which '——   ——' is true. Let us consider a second illustration.

2. The German mathematician Goldbach conjectured in 1742 that every even number (except 2, which, of course, is itself a prime) could be represented as the sum of two primes; 4, for example, may be represented as the sum of 2 and 2, 6 as the sum of 3 and 3, 8 as the sum of 5 and 3, and so on. Goldbach did not succeed in demonstrating his conjecture and handed it over to the famous mathematician Euler. Euler failed in turn to demonstrate it and the conjecture has remained ever since a brain-

[15] Intuitionist logic originated at the beginning of the twentieth century with the Dutch mathematician L. E. J. Brouwer; it was subsequently formalized by one of his disciples, A. Heyting.

teaser for number-theorists. In the late nineteen thirties, the Russian mathematician Vinogradoff obtained a result close to Goldbach's conjecture. He demonstrated the statement:

There exists an integer x such that all integers larger than x can be represented as sums of at most 4 primes (2),

by arguing roughly as follows: If the denial of (2) is true, a contradiction, which need not be specified here, follows. But a statement which logically implies a contradiction is itself false. The denial of (2) is therefore false, and hence (2) is true. This argument establishes for classical logic the existence of the integer x in question, even though it fails to exhibit x as being either this or that integer. For intuitionist logic, the argument establishes that the denial of (2) is false, but since it fails to exhibit x as being either this or that integer, it fails to establish the existence of x. The classical deduction of '(Ex)(—— x ——)' from '$\sim\sim$(Ex)(—— x ——)' is accordingly condemned by intuitionist logic. We have here an example of a sententially valid schema:

$$\sim\sim p \supset p,$$

which intuitionist logic must renounce because of its peculiar understanding of the quantifier '(Ex)'. Let us consider a third illustration.

3. We may first admit the following two definitions:

(a) A real number x is said to be *algebraic* when it satisfies some algebraic equation of the form:

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0,$$

where $n > 0$ and each a_k ($k = 1, 2, 3, \ldots, n$) is an integer larger than 0.

(b) A set is said to be *denumerable* when its members can be arranged in a sequence like the sequence of the integers: 1, 2, 3, ...

In order to demonstrate the statement:

There are real numbers which are not algebraic (3),

the German mathematician G. Cantor argued roughly as follows: If all real numbers were algebraic, the set of all real numbers would be denumerable; but the set of all real numbers is not denumerable; not all real numbers are therefore algebraic, and hence some real numbers are not algebraic. This argument establishes for classical logic the existence of non-algebraic or so-called *transcendental numbers*, even if it fails to provide means of constructing one. For intuitionist logic, the argument establishes that (3) is false, but since it fails to provide means of constructing any transcendental number, it fails to establish the existence

of transcendental numbers. The classical deduction of '(Ex)$\sim$(—— x ——)' from '$\sim(x)$(—— x ——)' is accordingly condemned by intuitionist logic. We have here an example of a quantificationally valid schema:

$$\sim(x)F(x) \supset (\mathrm{E}x)\sim F(x),$$

which intuitionist logic must renounce because of its peculiar understanding of the quantifier '(Ex)'.

Intuitionist logic consists of exactly the same signs and the same schemata as classical logic; all the schemata which it acknowledges as valid, we may call them here *I-valid*, are valid in classical logic, but some of the schemata which classical logic acknowledges as valid, we may call them here *C-valid*, are not valid in intuitionist logic.

Among the sentential schemata listed in section 7 which are not I-valid, we may quote:

T1: $p \vee \sim p$;
T5: $p \equiv \sim\sim p$;
T16a: $(p \supset q) \equiv (\sim q \supset \sim p)$;
T16b: $(p \equiv q) \equiv (\sim q \equiv \sim p)$;
T18: $(p \supset q) \equiv (\sim p \vee q)$;
T19: $(p \supset q) \equiv \sim(p \,.\, \sim q)$;

and

T21a: $\sim(p \,.\, q) \equiv (\sim p \vee q)$.

Half of T5 is I-valid, namely: '$p \supset \sim\sim p$', and '$\sim p \equiv \sim\sim\sim p$' is I-valid; all chains of negation signs thus reduce in intuitionist logic to '$\sim$' and '$\sim\sim$'. Half of T16a and half of T16b, namely: '$(p \supset q) \supset (\sim q \supset \sim p)$' and '$(p \equiv q) \supset (\sim q \equiv \sim p)$', are I-valid; of the two further laws of Transposition which are C-valid, '$(p \supset \sim q) \equiv (q \supset \sim p)$' and '$(\sim p \supset q) \equiv (\sim q \supset p)$', the first one is I-valid, but the second is not. Half of T18 and half of T19, namely: '$(\sim p \vee q) \supset (p \supset q)$' and '$(p \supset q) \supset \sim(p \,.\, \sim q)$', are I-valid. Finally, half of T21a, namely: '$(\sim p \vee \sim q) \supset \sim(p \,.\, q)$', is I-valid; T21b, '$\sim(p \vee q) \equiv (\sim p \,.\, \sim q)$', holds in intuitionist as well as in classical logic.

The converse halves of T16a, T16b, T18, T19, and T21a hold in intuitionist logic under the following guises:

(*a*) $(p \vee \sim p) \supset ((\sim q \supset \sim p) \supset (p \supset q))$,
$(p \vee \sim p) \supset ((\sim q \equiv \sim p) \supset (p \equiv q))$,
$(p \vee \sim p) \supset ((p \supset q) \supset (\sim p \vee q))$,
$(p \vee \sim p) \supset (\sim(p \,.\, \sim q) \supset (p \supset q))$,
$(p \vee \sim p) \supset (\sim(p \,.\, q) \supset (\sim p \vee \sim q))$;

(*b*) $(\sim\sim p \supset p) \supset ((\sim q \supset \sim p) \supset (p \supset q))$,
$(\sim\sim p \supset p) \supset ((\sim q \equiv \sim p) \supset (p \equiv q))$,
$(\sim\sim p \supset p) \supset ((p \supset q) \supset (\sim p \vee q))$,
$(\sim\sim p \supset p) \supset (\sim(p \,.\, \sim q) \supset (p \supset q))$,
$(\sim\sim p \supset p) \supset (\sim(p \,.\, q) \supset (\sim p \vee \sim q))$;

and

(*c*) $(\sim q \supset \sim p) \supset \sim\sim(p \supset q)$,
$(\sim q \equiv \sim p) \supset \sim\sim(p \equiv q)$,
$(p \supset q) \supset \sim\sim(\sim p \vee q)$,
$\sim(p \,.\, \sim q) \supset \sim\sim(p \supset q)$,
$\sim(p \,.\, q) \supset \sim\sim(\sim p \vee \sim q)$.

Most of the remaining sentential schemata listed in section 7 are I-valid.

Among the quantificational schemata listed in section 17 which are not I-valid, we may quote:

the following three laws of Opposition:

Q30a: $\sim(x)F(x) \equiv (\mathrm{E}x)\sim F(x)$,
Q30c: $(x)F(x) \equiv \sim(\mathrm{E}x)\sim F(x)$,
Q30d: $(\mathrm{E}x)F(x) \equiv \sim(x)\sim F(x)$;

the following law of Distributivity:

Q36c: $(\mathrm{E}x)(F(x) \supset G(x)) \equiv ((x)F(x) \supset (\mathrm{E}x)G(x))$;

and the following four laws of Confinement:

Q37e: $(x)(p \vee F(x)) \equiv (p \vee (x)F(x))$,
Q37f: $(x)(F(x) \vee p) \equiv ((x)F(x) \vee p)$,
Q37k: $(\mathrm{E}x)(p \supset F(x)) \equiv (p \supset (\mathrm{E}x)F(x))$,
Q37l: $(\mathrm{E}x)(F(x) \supset p) \equiv ((x)F(x) \supset p)$.

The following conditionals, however, are I-valid:

$(\mathrm{E}x)\sim F(x) \supset \sim(x)F(x)$,
$(x)F(x) \supset \sim(\mathrm{E}x)\sim F(x)$,
$(\mathrm{E}x)F(x) \supset \sim(x)\sim F(x)$,
$(\mathrm{E}x)(F(x) \supset G(x)) \supset ((x)F(x) \supset (\mathrm{E}x)G(x))$,
$(p \vee (x)F(x)) \supset (x)(p \vee F(x))$,
$((x)F(x) \vee p) \supset (x)(F(x) \vee p)$,
$(\mathrm{E}x)(p \supset F(x)) \supset (p \supset (\mathrm{E}x)F(x))$,

and

$(\mathrm{E}x)(F(x) \supset p) \supset ((x)F(x) \supset p)$.

Most of the remaining quantificational schemata listed in section 17 are I-valid.

Deductions may be carried out in intuitionist logic with the help of the familiar rules of Detachment, Adjunction, Interchange, Relettering, and Universalization; the rules of Sentential and Quantificational Insertion must, however, be restricted to instances of recorded I-valid schemata. A deduction set up in accordance with these six rules is sometimes called *constructive.* Attempts have been made during the last four decades to replace all the non-constructive deductions of mathematics by constructive ones. Success has been achieved in many cases; the above deduction of the existence of transcendental numbers, for example, can be replaced by a constructive one, also outlined by Cantor. But some non-constructive deductions, Vinogradoff's for example, have not been replaced yet by constructive ones, and others are known not to be replaceable by constructive ones.

The intuitionist criteria of truth and existence have been offered as criteria of mathematical truth and mathematical existence only; their advocate, Brouwer, declined to turn them into general criteria of truth and existence.

CHAPTER THREE

Formalization of Sentential and Quantificational Logic

22. INTRODUCTION

In the previous chapters we studied sentential logic and quantificational logic as language-forms filled out with factual arguments, predicates, and statements. In the present chapter we shall ignore the factual signs in question and reconstrue both logics as pure language-forms.

In the previous two chapters we also studied sentential logic and quantificational logic as systems of interpreted signs. In the present chapter we shall ignore the interpretation placed so far upon connectives, quantifiers, and so on, and reconstrue both logics as systems of uninterpreted signs, or, to use the technical phrase, as *calculi*. Our reconstruction, though syntactical in nature, will often be guided, however, by semantical and pragmatical motives.

23. PRIMITIVE AND DEFINED SIGNS

A calculus consists of a (finite or infinite) number of signs which may enter as components in what we call *formulae*, but are not analyzable themselves into further components. The sentential calculus, for instance, will consist of:

(*a*) an infinite set of sentential dummies: 'p', 'q', 'r', 's', 'p'', 'q'', 'r'', 's'', and so on;
(*b*) nine connectives: '$\sim$', '$\vee$', '$.$', '$\supset$', '$\subset$', '$\equiv$', '$\not\equiv$', '$|$', and '$\downarrow$';
(*c*) two parentheses: '(' and ')'.

The quantificational calculus, on the other hand, will consist of the same signs plus:

(*d*) an infinite set of predicate dummies: 'F', 'G', 'H', 'F'', 'G'', 'H'', and so on;
(*e*) an infinite set of argument variables: 'w', 'x', 'y', 'z', 'w'', 'x'', 'y'', 'z'', and so on;

(*f*) one quantifier letter: 'E';
(*g*) one comma: ','.

We may include all the signs listed in (*a*)–(*c*) within the basic vocabulary of the sentential calculus, or we may exclude some of them from that vocabulary and bring them back into the calculus through so-called *definitions*. Similarly, we may include all the signs listed in (*a*)–(*g*) within the basic vocabulary of the quantificational calculus, or we may exclude some of them from that vocabulary and bring them back into the calculus through definitions. If the second strategy is adopted in each case, then the signs of our calculi will fall into two sets, a set of so-called *primitive signs* and a set of so-called *defined signs*.

By reducing the number of primitive signs of a calculus C and enlarging the number of its defined signs we simplify, on one hand, the structure of C without, on the other, weakening its means of expression. We shall accordingly make it a rule, when formalizing a calculus C, to select as the primitive signs of C a set of signs no member of which is definable in terms of any other member.[1] We shall, for instance, select as primitive signs of the sentential calculus the sentential dummies listed in (*a*), the two connectives '$\sim$' and 'v', and the two parentheses '(' and ')', because none of these signs is definable in terms of any other primitive sign of the calculus; we shall, however, treat the seven connectives '.', '$\supset$', '$\subset$', '$\equiv$', '$\not\equiv$', '|', and '$\downarrow$', as defined signs, each one of them being definable in terms of the above mentioned primitives. We shall next select as primitive signs of the quantificational calculus the sentential dummies listed in (*a*), the predicate dummies listed in (*d*), the argument variables listed in (*e*), the two connectives '$\sim$' and 'v', the two parentheses '(' and ')', and the comma ',', because none of these signs is definable in terms of any other primitive sign of the calculus; we shall, however, treat the seven connectives: '.', '$\supset$', '$\subset$', '$\equiv$', '$\not\equiv$', '|', and '$\downarrow$', and the quantifier letter 'E' as defined signs, each one of them being definable in terms of the above mentioned primitives.

The crucial question next arises: "Which signs of a given calculus C are definable in terms of which?" Before facing it, let us say a word on definitions.

Definitions are metalogical prescriptions to the effect that a formula or a set of formulae, called *the definiendum*, may in a given calculus C serve as a paraphrase for another formula or another set of formulae, called *the definiens*. The *definiendum* includes among, possibly, other signs (either primitive or predefined ones) the sign to be defined; the *definiens*, on the other hand, consists exclusively of primitive or predefined signs.

[1] We shall depart from this rule once in section 31.

As a sample we may quote the classical definition of '.' in terms of '$\sim$' and '$\vee$', namely:

$$\ulcorner(\varphi \,.\, \psi)\urcorner \rightarrow \ulcorner\sim(\sim\varphi \vee \sim\psi)\urcorner \qquad (1),$$

where the arrow '$\rightarrow$' reads: 'is defined as'. (1) is a metalogical prescription to the effect that formulae of the form $\ulcorner(\varphi \,.\, \psi)\urcorner$ may serve within the sentential calculus as paraphrases for formulae of the form $\ulcorner\sim(\sim\varphi \vee \sim\psi)\urcorner$;

$$\ulcorner(\varphi \,.\, \psi)\urcorner,$$

the *definiendum*, includes the sign '.' to be defined along with two primitive signs of the calculus (the left-hand parenthesis and the right-hand one) and two names of unspecified formulae of the calculus (the Greek letters 'φ' and 'ψ');

$$\ulcorner\sim(\sim\varphi \vee \sim\psi)\urcorner,$$

the *definiens*, consists of four primitive signs of the calculus (the two parentheses '(' and ')' and the two connectives '$\sim$' and '$\vee$') plus the two Greek letters 'φ' and 'ψ'.

The question we raised above will now read: "Which formula φ may in a given calculus C serve as a *definiens* for another formula ψ?" Two answers have been given to it:

(*a*) a *pragmatical* one according to which a formula φ may in a given calculus C serve as a *definiens* for another formula ψ if and only if φ and ψ, once interpreted, have the same meaning or, to use a technical phrase, *are synonymous*;

(*b*) a *semantical* one according to which a formula φ may in a given calculus C serve as a *definiens* for another formula ψ if and only if φ and ψ, once interpreted, may be interchanged in any formula χ of C without altering the truth-values of χ.

Criterion (*b*) is weaker than criterion (*a*) and handier; we take it as our official criterion of definability. In accordance with it, we shall use (1) as our definition of '.'; the biconditional:

$$\ulcorner(\varphi \,.\, \psi) \equiv \sim(\sim\varphi \vee \sim\psi)\urcorner,$$

being sententially valid, the two formulae $\ulcorner(\varphi \,.\, \psi)\urcorner$ and $\ulcorner\sim(\sim\varphi \vee \sim\psi)\urcorner$ have the same truth-values and hence may be interchanged in any sentential or quantificational formula χ without altering the truth-values of χ. Similarly, we shall use:

$$\ulcorner(\varphi \supset \psi)\urcorner \rightarrow \ulcorner(\sim\varphi \vee \psi)\urcorner,$$
$$\ulcorner(\varphi \subset \psi)\urcorner \rightarrow \ulcorner(\psi \supset \varphi)\urcorner,$$
$$\ulcorner(\varphi \equiv \psi)\urcorner \rightarrow \ulcorner((\varphi \supset \psi) \,.\, (\psi \supset \varphi))\urcorner,$$
$$\ulcorner(\varphi \not\equiv \psi)\urcorner \rightarrow \ulcorner\sim(\varphi \equiv \psi)\urcorner,$$
$$\ulcorner(\varphi \mid \psi)\urcorner \rightarrow \ulcorner(\sim\varphi \vee \sim\psi)\urcorner,$$

and

$$\ulcorner(\varphi \downarrow \psi)\urcorner \rightarrow \ulcorner(\sim\varphi \,.\, \sim\psi)\urcorner,$$

as our definitions of '$\supset$', '$\subset$', '$\equiv$', '$\not\equiv$', '$|$', and '$\downarrow$', respectively. The biconditionals:

$$\ulcorner(\varphi \supset \psi) \equiv (\sim\varphi \vee \psi)\urcorner,$$
$$\ulcorner(\varphi \subset \psi) \equiv (\psi \supset \varphi)\urcorner,$$
$$\ulcorner(\varphi \equiv \psi) \equiv ((\varphi \supset \psi) \,.\, (\psi \supset \varphi))\urcorner,$$
$$\ulcorner(\varphi \not\equiv \psi) \equiv \sim(\varphi \equiv \psi)\urcorner,$$
$$\ulcorner(\varphi \mid \psi) \equiv (\sim\varphi \vee \sim\psi)\urcorner,$$

and

$$\ulcorner(\varphi \downarrow \psi) \equiv (\sim\varphi \,.\, \sim\psi)\urcorner,$$

being sententially valid, the six *definientia* in question have the same truth-values as their respective *definienda* and hence may be interchanged with them in any sentential or quantificational formula χ without altering the truth-values of χ.

We shall finally use:

$$\ulcorner(\text{E} \quad)\varphi\urcorner \rightarrow \ulcorner\sim(\quad)\sim\varphi\urcorner,$$

with its two blanks filled by one argument variable, as our definition of 'E'. The biconditional:

$$\ulcorner(\text{E} \quad)\varphi \equiv \sim(\quad)\sim\varphi\urcorner$$

(with its two blanks filled by one argument variable) being quantificationally valid, the two formulae $\ulcorner(\text{E} \quad)\varphi\urcorner$ and $\ulcorner\sim(\quad)\sim\varphi\urcorner$ (with their respective blanks filled by one argument variable) have the same truth-values and hence may be interchanged in any quantificational formula χ without altering the truth-values of χ.

The above definitions define the eight signs '.', '$\supset$', '$\subset$', '$\equiv$', '$\not\equiv$', '$|$', '$\downarrow$', and 'E' in a context;

$$\ulcorner(\varphi \,.\, \psi)\urcorner \rightarrow \ulcorner\sim(\sim\varphi \vee \sim\psi)\urcorner,$$

for instance, defines the connective '.' in the context $\ulcorner(\varphi \qquad \psi)\urcorner$, and

$$\ulcorner(\text{E} \quad)\varphi\urcorner \rightarrow \ulcorner\sim(\quad)\sim\varphi\urcorner,$$

defines the quantifier letter 'E' in the context $\ulcorner(\qquad)\varphi\urcorner$. They are accordingly called *contextual definitions. Non-contextual* definitions which define a given sign out of any context will be used in chapter four to introduce two class signs and two relation signs into the calculus of identity.

Contextual definitions fall into two groups: *explicit* definitions like the above definitions of '.', '$\supset$', '$\subset$', '$\equiv$', '$\not\equiv$', '$|$', '$\downarrow$', and 'E'; and *non-explicit* or *recursive* definitions. An explicit definition defines a sign in

one step; a recursive definition, on the other hand, defines a sign in several steps: it first defines the sign within an initial context or set of contexts and next gives recipes for reducing other contexts showing the same sign to the initial ones. A recursive definition will be used in chapter four to introduce into the calculus of identity the so-called *numerical quantifiers* 'There exist at least n x such that $F(x)$', where $n = 1, 2, 3, \ldots$. It will roughly read as follows:

'There exists at least 1 x such that $F(x)$' $\rightarrow$ '$(\mathrm{E}x)F(x)$' (2),

'There exist at least $n + 1$ x such that $F(x)$' $\rightarrow$ '$(\mathrm{E}x)(F(x)$. (There exist at least n y such that $F(y)$. $y \neq x$))' (3).[2]

(2) paraphrases the initial context 'There exists at least 1 x such that $F(x)$' as '$(\mathrm{E}x)F(x)$'; (3) gives a recipe for reducing any context of the form 'There exist at least $n + 1$ x such that $F(x)$' to a context of the form 'There exist at least n y such that $F(y)$'.

To see how the definition works, let us take n to be 2; then

There exist at least 2 x such that $F(x)$

becomes by (3):

$(\mathrm{E}x)(F(x)$. (There exists at least 1 y such that $F(y)$. $y \neq x$)),

which becomes by (2):

$$(\mathrm{E}x)(F(x)\ .\ (\mathrm{E}y)(F(y)\ .\ y \neq x)).$$

Definitions have been exclusively viewed so far as devices for introducing non-primitive signs into what we called *calculi*; but they may also serve to introduce various phrases and, especially, various predicates into the syntax, semantics, or pragmatics of a given calculus. In the following pages, for example, we shall define such syntactical predicates as 'is a formula of the sentential calculus', 'is a well-formed formula of the sentential calculus', and so on; these definitions will introduce into our syntax language, English, a number of technical phrases we need to discourse about the sentential calculus. It would of course be possible to formalize in turn the syntax of the sentential calculus as a calculus, to single out certain English words as syntactical primitives, and introduce the predicates 'is a formula of the sentential calculus', 'is a well-formed formula of the sentential calculus', and so on, as paraphrases of appropriate sequences of syntactical primitives. A syntax of this sort is usually called a *formal syntax*; ours will be an *informal* one.[3]

[2]The definition, phrased for convenience's sake with 'x's, 'y's, and 'F's, will be rephrased below in metalogical style.

[3]The reader will find a sample formal syntax in Quine's *Mathematical Logic,* chapter seven.

Before closing this section, we may insert three remarks on the choice of our primitive signs:

1. The primitive signs of a given calculus C must allow for the definition of all the non-primitive signs of C. This requirement is satisfied here since all of '$.$', '$\supset$', '$\subset$', '$\equiv$', '$\not\equiv$', '$|$', '$\downarrow$', and 'E' are definable in terms of our primitive signs.

2. A given calculus C may contain several sets of signs which, if adopted as primitive, would satisfy requirement 1. We may, for instance, adopt '$\sim$' and '$.$' rather than '$\sim$' and '$\vee$' as primitive connectives of the sentential calculus and define all of '$\vee$', '$\supset$', '$\subset$', '$\equiv$', '$\not\equiv$', '$|$', '$\downarrow$' in terms of them; we need only define $\ulcorner(\varphi \vee \psi)\urcorner$ as $\ulcorner\sim(\sim\varphi . \sim\psi)\urcorner$ and carry over the original definitions of '$\supset$', '$\subset$', '$\equiv$', '$\not\equiv$', '$|$', and '$\downarrow$'. Other selections of primitive connectives satisfying requirement 1 are still possible like:

(*a*) '$\sim$' and '$\supset$';
(*b*) '$\sim$' and '$\subset$';
(*c*) '$|$';
(*e*) '$\downarrow$', and so on.

3. In view of 2 a sign which in a given version of a calculus C figures as primitive may in a different version of the same calculus figure as a defined sign; such is the case with '$\vee$', primitive in our version of the sentential calculus, but defined in the five alternative versions sketched in 2.

24. FORMULAE AND WELL-FORMED FORMULAE

The primitive signs of a calculus C may be aligned in finite sequences called *formulae of C*. These formulae usually fall, depending upon the way in which their components are aligned, into two classes: a class of *well-formed* formulae and a class of *non-well-formed* formulae. The formulae of the sentential and of the quantificational calculus, for instance, will fall into two such classes; the first will include formulae like '$(p \vee q)$', '$F(x,y)$', '$(x)(\sim F(y) \vee p)$', and so on; the second will include formulae like '$p \vee$', '$F(x,q)$', '$(x)(\sim F(y) \vee))$', and so on. The distinction between well-formed and non-well-formed formulae is purely syntactical; it usually matches, however, a pragmatical distinction, the distinction between meaningful and meaningless formulae. We may accordingly expect the well-formed formulae of a calculus to become, once interpreted, meaningful items of discourse; its non-well-formed formulae, meaningless ones.

The well-formed formulae of a calculus C are usually isolated through a recursive definition which: (*a*) pronounces an initial set of formulae of

C well-formed and (*b*) pronounces well-formed certain compounds of well-formed formulae of C.

The definition of a well-formed formula of the sentential calculus, for instance, will read:

1. A sentential dummy is well-formed;
2. If a formula φ is well-formed, so is the formula $\ulcorner \sim\varphi \urcorner$;
3. If two formulae φ and ψ are well-formed, so is the formula $\ulcorner(\varphi \vee \psi)\urcorner$.

Step 1 pronounces an initial set of formulae well-formed, the sentential dummies 'p', 'q', 'r', 's', and so on; steps 2 and 3 next pronounce well-formed all negations and inclusive alternations whose components are themselves well-formed.[4]

Consider, for instance, the formula '$(\sim p \vee (q \vee r))$'. According to 3 '$(\sim p \vee (q \vee r))$' is well-formed if its two components '$\sim p$' and '$(q \vee r)$' are well-formed; according to 2 '$\sim p$' is well-formed if its component 'p' is well-formed, and according to 3 '$(q \vee r)$' is well-formed if its two components 'q' and 'r' are well-formed; but according to 1 'p', 'q', and 'r' are well-formed; the formula '$(\sim p \vee (q \vee r))$' is therefore well-formed. Consider next the formula '$(\sim p \vee))$'. According to 3 '$(\sim p \vee))$' is well-formed if its two components '$\sim p$' and ')' are well-formed; '$\sim p$', as we just noted, is well-formed, but ')', though a formula, is not a well-formed formula; '$(\sim p \vee))$' is therefore not well-formed. As these two examples may suggest, steps 1–3 enable us to check mechanically whether any given formula of the sentential calculus is well-formed or not.

An expression like '$(p \,.\, q)$' or '$(p \supset q)$' in which occurs a defined sign of the sentential calculus is not a well-formed formula of that calculus; it is, at best, a paraphrase of one. For convenience's sake, however, we shall stretch the label 'well-formed formula' to cover both well-formed formulae and paraphrases of well-formed formulae. '$(p \,.\, q)$' and '$(p \supset q)$', being paraphrases of the two well-formed formulae '$\sim(p \vee \sim q)$' and '$(\sim p \vee q)$', respectively, will thus count as well-formed formulae of the sentential calculus.

The definition of a well-formed formula of the quantificational calculus will roughly read as follows:

1. A sentential dummy is well-formed;
2. The result of filling the first blank of ' (, . . . ,) ' with a predicate dummy and its following blanks with one or more argument variables is well-formed;
3. If a formula φ is well-formed, so is the result of filling the blank of $\ulcorner(\quad)\varphi\urcorner$ with an argument variable;

[4]Well-formed formulae of types 2 and 3 correspond to the sentential schemata of chapter one.

4. If a formula φ is well-formed, so is the formula $\ulcorner\sim\varphi\urcorner$;
5. If two formulae φ and ψ are well-formed, so is the formula $\ulcorner(\varphi \vee \psi)\urcorner$.

Steps 1 and 2 here pronounce two initial sets of formulae well-formed; steps 3, 4, and 5 next pronounce well-formed all negations, inclusive alternations, and universally quantified formulae whose components are themselves well-formed.[4a]

Consider, for instance, the formula '$(x)(\sim F(x) \vee p)$'. According to 3 '$(x)(\sim F(x) \vee p)$' is well-formed if its component '$(\sim F(x) \vee p)$' is well-formed; according to 5 '$(\sim F(x) \vee p)$' is well-formed if its two components '$\sim F(x)$' and 'p' are well-formed; according to 4 '$\sim F(x)$' is well-formed if its component '$F(x)$' is well-formed; but, according to 1, 'p' is well-formed and, according to 2, '$F(x)$' is well-formed; '$(x)(\sim F(x) \vee p)$' is therefore well-formed. Consider next the formula '$\sim(x)F(x,q)$'. According to 4 '$\sim(x)F(x,q)$' is well-formed if its component '$(x)F(x,q)$' is well-formed; according to 3 '$(x)F(x,q)$' is well-formed if its component '$F(x,q)$' is well-formed; but '$F(x,q)$' is not well-formed; '$\sim(x)F(x,q)$' is therefore not well-formed. As these two examples may suggest, steps 1–5 enable us to check mechanically whether any given formula of the quantificational calculus is well-formed or not.

An expression like '$(\mathrm{E}x)F(x)$' or '$(\mathrm{E}x)(F(x) \supset p)$' in which occurs a defined sign of the quantificational calculus is not a well-formed formula of that calculus; it is, at best, a paraphrase of one. For convenience's sake, however, we shall stretch the label 'well-formed formula' to cover both well-formed formulae and paraphrases of well-formed formulae. '$(\mathrm{E}x)F(x)$' and '$(\mathrm{E}x)(F(x) \supset p)$', being paraphrases of the two well-formed formulae '$\sim(x)\sim F(x)$' and '$\sim(x)\sim(\sim F(x) \vee p)$', respectively, will thus count as well-formed formulae of the quantificational calculus.

A glance back should convince the reader that if the two connectives '$\sim$' and '$\vee$' and the universal quantifiers '(w)', '(x)', '(y)', '(z)', and so on, are given their intended meanings, then all the formulae which steps 1–5 identify as well-formed are meaningful formulae.[5]

25. AXIOMS, RULES OF DEDUCTION, AND THEOREMS

As the formulae of a calculus C fall into two groups: well-formed and non-well-formed formulae, so its well-formed formulae may fall into two

[4a] Well-formed formulae of type 2 correspond to the atomic quantificational schemata of chapter two; well-formed formulae of type 3 correspond to the quantified schemata of chapter two; and well-formed formulae of types 4 and 5 correspond to the sentential schemata of chapter one and the molecular quantificational schemata of chapter two.

[5] The well-formed formulae '$(x)p$', '$(x)F(y)$', and so on, over which the reader might stumble, may respectively be treated as redundant versions of 'p', '$F(y)$' and so on.

subgroups: *theorems* and *non-theorems.* The distinction between theorems and non-theorems is purely syntactical; it is meant, however, to match a semantical distinction, the distinction between valid and non-valid formulae.

The theorems of a calculus C are usually isolated through a method reminiscent of the methods of section 8 and section 18: *the deductive method.* It works as follows:

1. Certain formulae of C are first appointed as *axioms of* C;
2. Certain metalogical rules are next selected as *rules of deduction in* C;
3. Further formulae of C are then deduced from the axioms of C with the help of the rule of deduction in C and labelled *theorems of* C.

The deductive method was first used by Euclid in his famous *Elements of Geometry* (*circa* 300 B. C.). The feeling long prevailed that the method was proper to geometry, but Frege showed in 1879 that it may be applied to logic and some ten years later the Italian mathematician Peano showed that it may be applied to arithmetic. Its applicability to the natural sciences has recently been illustrated by various studies in physics, biology, and psychology.

In keeping with the above program we shall, in a first version of the sentential calculus, appoint as axioms the following four formulae, numbered A100–A103:

A100: $(p \vee p) \supset p$,
A101: $p \supset (p \vee q)$,
A102: $(p \vee q) \supset (q \vee p)$,
A103: $(p \supset q) \supset ((r \vee p) \supset (r \vee q))$;

and select as rules of sentential deduction the familiar rule of Detachment and a rule of substitution. In a second version of the calculus we shall replace A100–A103 by four infinite sets of axioms (to which, by the way, will belong A100–A103) and drop the rule of Substitution made superfluous by our new axioms.

In both versions we shall define a *sentential proof* as a finite sequence of well-formed formulae each one of which is a sentential axiom or is obtainable from previous formulae in the sequence through application of a rule of sentential deduction, and define a *sentential theorem* as the last formula of a sentential proof. We include as a sample a proof borrowed from our first version of the sentential calculus:

(1): $(p \supset q) \supset ((r \vee p) \supset (r \vee q))$,
(2): $((p \vee q) \supset (q \vee p)) \supset ((\sim p \vee (p \vee q)) \supset (\sim p \vee (q \vee p)))$,
(3): $(p \vee q) \supset (q \vee p)$,

(4): $(\sim p \vee (p \vee q)) \supset (\sim p \vee (q \vee p))$,
(5): $\sim p \vee (p \vee q)$,
(6): $\sim p \vee (q \vee p)$.

This sequence consists of three axioms and three formulae obtainable from previous formulae in the sequence through application of a rule of deduction; formulae (1), (3), and (5) are axioms;[6] formula (2) is obtainable from formula (1) through application of the rule of Substitution; formula (4) is obtainable from formulae (2) and (3) through application of the rule of Detachment; and formula (6) is obtainable from formulae (4) and (5) through application again of the rule of Detachment. Formula (6), the last formula in the sequence, may accordingly be pronounced a sentential theorem.

Similarly in a first version of the quantificational calculus we shall appoint as axioms the above four formulae A100–A103 plus the two quantificational formulae:

A200: $(x)F(x) \supset F(y)$,
A201: $(x)(p \supset F(x)) \supset (p \supset (x)F(x))$;

and select as rules of quantificational deduction the familiar rules of Detachment, Universalization, and Relettering, and three rules of Substitution. In a second version of the calculus we shall replace A100–A103 and A200–A201 by six infinite sets of axioms (to which will belong A100–A103 and A200–A201) and drop the rule of Relettering and the three rules of Substitution made superfluous by our new axioms.

In both versions we shall define a *quantificational proof* as a finite sequence of well-formed formulae each one of which is a quantificational axiom or is obtainable from previous formulae in the sequence through application of a rule of quantificational deduction, and define a *quantificational theorem* as the last formula of a quantificational proof.[7]

It can be shown, and we shall do so in chapter five, that a formula is provable as a sentential or a quantificational theorem if and only if it is sententially or quantificationally valid. The deductive method thus isolates, as expected, the valid formulae of our two calculi; it does not isolate them, however, mechanically. Our definition of a proof enables us to check mechanically whether any given sequence of well-formed formulae is a proof or not; our definition of a theorem does not enable us,

[6](1) is A103, (3) is A102, and (5) is, by definition, a paraphrase of A101; defined signs may be introduced into or eliminated from a formula at will.

[7]Along with proofs we shall introduce in sections 28 and 29 so-called *derivations from assumption formulae* which match the sentential and quantificational deductions of chapter one and chapter two.

however, to check mechanically whether any given formula is a theorem or not. Given a proof we may indeed check mechanically whether a given formula is its last formula or not; but given a formula we cannot check mechanically whether it is the last formula of a proof or not. It is only in the few calculi which, like the sentential calculus or the monadic quantificational calculus, have a decision procedure that recipes may be given for the mechanical construction of proofs and hence the mechanical identification of theorems. In all other calculi proofs have to be discovered by hit or miss, and the identification of theorems has to await the discovery of proofs of which they happen to be the last formulae.

In the following sections we shall prove a number of theorems of the sentential and of the quantificational calculus; but we shall also prove a number of theorems about the sentential and the quantificational calculus. To prevent confusion we shall call the latter theorems *metatheorems*. Typical as metatheorems are the following:

Metatheorem A: Every well-formed formula of the sentential calculus contains at least one sentential dummy,

(which we shall not bother to prove), or the following:

Metatheorem B: Every well-formed formula of the form

$$\ulcorner \varphi \supset (\psi \supset \varphi) \urcorner$$

is a theorem of the sentential calculus,

(which we shall prove in section 28), or the following:

Metatheorem C: Every valid formula of the sentential calculus is provable as a theorem of the sentential calculus,

(which we shall prove in chapter five).

The proofs of these metatheorems will differ considerably from the proof of the theorem '$\sim p \vee (q \vee p)$' displayed above. They will belong to the syntax or the semantics of our two calculi and hence be couched in English; they will use whatever evidence may be gathered about our two calculi, whether that evidence be factual or mathematical; they will finally be informal proofs. It would of course be possible:

1. to formalize the syntax or semantics of our two calculi as a new calculus;
2. to appoint a few syntactical or semantical formulae as axioms;
3. to select a few meta-metalogical rules as rules of deduction;

and 4. to obtain all our metatheorems as terminating formulae of formal proofs.

A syntax or semantics of this sort is usually called an *axiomatic syntax* or *axiomatic semantics*; ours will be a non-axiomatic one.[8]

We shall close this section with a few words on the selection of our axioms.

1. The formulae selected as axioms of a given calculus C must allow for the deduction of all the valid formulae of C. This requirement is satisfied, as noted above, by our axioms.

2. A calculus C may contain several sets of formulae which, if selected as axioms, would satisfy requirement 1. We shall in section 30 list a number of such sets for our two calculi.

3. In view of 2 a formula which in a given version of a calculus C figures as an axiom may in a different version of the same calculus figure as a theorem; such is the case with some of the formulae A100–A103 and A200–A201, axioms in our version of the sentential or of the quantificarional calculus, but theorems in the alternative versions referred to in 2.

26. SYNTACTICAL NOTATIONS

We appoint the following four sets of metalogical variables:

1. the four Greek letters 'φ', 'χ', 'ψ', and 'ω', to take sentential or quantificational formulae as their values;
2. the four Greek letters 'α', 'β', 'γ', and 'δ', to take argument variables as their values;
3. the four Greek letters 'ζ', 'η', 'θ', and 'ι', to take sentential dummies as their values;
4. the four Greek letters 'κ', 'λ', 'μ', and 'ν', to take predicate dummies as their values.

These twelve letters will sometimes be used with accents and/or subscripts. The two corners '$\ulcorner$' and '$\urcorner$' will be used as in chapter one and chapter two.

27. THE SENTENTIAL CALCULUS, VERSION I

1. *Primitive signs.* The primitive signs of the sentential calculus are:

(*a*) the infinite set of sentential dummies: 'p', 'q', 'r', 's', 'p'', 'q'', 'r'', 's'', 'p''', 'q''', 'r''', 's''', and so on;

(*b*) the two connectives '$\sim$' and '$\vee$';

(*c*) the two parentheses '(' and ')'.

2. *Definition of a formula*: A formula is a finite sequence of primitive signs.

[8]The reader will find a sample axiomatic syntax and a sample axiomatic semantics in Tarski's *Logic, Semantics, Metamathematics.*

3. *Definition of a well-formed formula*:

(*a*) A sentential dummy is a well-formed formula;

(*b*) If φ is a well-formed formula, so is $\ulcorner\sim\varphi\urcorner$;

(*c*) If φ and ψ are well-formed formulae, so is $\ulcorner(\varphi \vee \psi)\urcorner$.

4. *Defined signs*. The following seven definitions are adopted:

D1: $\ulcorner(\varphi \,.\, \psi)\urcorner \rightarrow \ulcorner\sim(\sim\varphi \vee \sim\psi)\urcorner$;
D2: $\ulcorner(\varphi \supset \psi)\urcorner \rightarrow \ulcorner(\sim\varphi \vee \psi)\urcorner$;
D3: $\ulcorner(\varphi \subset \psi)\urcorner \rightarrow \ulcorner(\psi \supset \varphi)\urcorner$;
D4: $\ulcorner(\varphi \equiv \psi)\urcorner \rightarrow \ulcorner((\varphi \supset \psi) \,.\, (\psi \supset \varphi))\urcorner$;
D5: $\ulcorner(\varphi \not\equiv \psi)\urcorner \rightarrow \ulcorner\sim(\varphi \equiv \psi)\urcorner$;
D6: $\ulcorner(\varphi \mid \psi)\urcorner \rightarrow \ulcorner(\sim\varphi \vee \sim\psi)\urcorner$;
D7: $\ulcorner(\varphi \downarrow \psi)\urcorner \rightarrow \ulcorner(\sim\varphi \,.\, \sim\psi)\urcorner$.

Note: In view of 3(*c*) and D1–D7, all binary compounds should be enclosed within parentheses; in practice, however, we shall omit such parentheses when the compounds in question do not stand within any further compounds.

5. *Axioms and rules of deduction*:

(*a*) Axioms:

A100: $(p \vee p) \supset p$;
A101: $p \supset (p \vee q)$;
A102: $(p \vee q) \supset (q \vee p)$;
A103: $(p \supset q) \supset ((r \vee p) \supset (r \vee q))$.[9]

(*b*) Rules of deduction:

R1: From φ and $\ulcorner\varphi \supset \psi\urcorner$ one may deduce ψ (Detachment).

R3a: Let $\varphi\frac{\psi}{\zeta}$ be the result of replacing at each one of its occurrences in φ the sentential dummy ζ by the well-formed formula ψ. From φ one may deduce $\varphi\frac{\psi}{\zeta}$ (Sentential Substitution).[10]

Note: R3a allows us to substitute for only one sentential dummy at a time. To simplify matters, however, we shall often substitute for several sentential dummies at a time and leave it to the reader to break our simultaneous substitutions into sequences of single substitutions. In step

[9] A100–A103 are an adaptation of Whitehead and Russell's axioms for the sentential calculus in *Principia Mathematica*; A101 and A103 were respectively recorded in chapter one as T7a and T15.

[10] R2, R3b, and R3c will be rules of quantificational deduction.

(1) of the proof of T108, for instance, we simultaneously substitute '$\sim p$' for 'p' and 'p' for 'q' in A102 to obtain:

$$(\sim p \vee p) \supset (p \vee \sim p),$$

and in step (1) of the proof of T105 we simultaneously substitute '$p \vee q$' for 'p', '$q \vee p$' for 'q', and '$\sim p$' for 'r' in A103 to obtain:

$$((p \vee q) \supset (q \vee p)) \supset ((\sim p \vee (p \vee q)) \supset (\sim p \vee (q \vee p))).$$

The former substitution is easily broken into a sequence of single substitutions, namely: substitution of '$\sim p$' for 'p' in A102 to obtain:

$$(\sim p \vee q) \supset (q \vee \sim p) \qquad (1),$$

and substitution of 'p' for 'q' in (1) to obtain:

$$(\sim p \vee p) \supset (p \vee \sim p).$$

The latter can be broken into a sequence of single substitutions as follows:

1. Substitute for 'p' in A103 a sentential dummy foreign to A103 to obtain, say,

$$(s \supset q) \supset ((r \vee s) \supset (r \vee q)) \qquad (2);$$

2. Substitute for 'q' in (2) a sentential dummy foreign to (2) to obtain, say,

$$(s \supset s') \supset ((r \vee s) \supset (r \vee s')) \qquad (3);$$

3. Substitute for 's' in (3) '$p \vee q$' to obtain:

$$((p \vee q) \supset s') \supset ((r \vee (p \vee q)) \supset (r \vee s')) \qquad (4);$$

4. Substitute for 's'' in (4) '$q \vee p$' to obtain:

$$((p \vee q) \supset (q \vee p)) \supset ((r \vee (p \vee q)) \supset (r \vee (q \vee p))) \qquad (5);$$

5. Substitute for 'r' in (5) '$\sim p$' to obtain:

$$((p \vee q) \supset (q \vee p)) \supset ((\sim p \vee (p \vee q)) \supset (\sim p \vee (q \vee p))).$$

6. *Definitions of a proof and of a theorem*: A proof is a finite sequence of well-formed formulae each one of which is an axiom or is obtainable from previous formulae in the sequence through Detachment or Substitution; a theorem is the last formula of a proof.

Note: Since a theorem can always be thought of as standing for its entire proof, we shall allow both axioms and theorems to figure in the sample proofs below.

7. *Conventions of proof recording.* We shall obey the following conventions when recording proofs:

(*a*) Each step of a proof will be written on a separate line.

(*b*) Each step of a proof will be preceded by a reference numeral in parentheses.

(*c*) Each step of a proof will be justified by the following data:

(*c*1) If a given step (n) follows through Detachment from two steps (l) and (m), we shall insert to the right of (n): '(D, l, m)' or, more simply, '(l, m)'.

(*c*2) If a given step (n) follows through Substitution from a step (m), we shall insert to the right of (n): '$(m, \psi_1/\zeta_1, \psi_2/\zeta_2, \ldots, \psi_n/\zeta_n)$', where $\psi_1, \psi_2, \ldots, \psi_n$ are the formulae respectively substituted for the sentential dummies $\zeta_1, \zeta_2, \ldots, \zeta_n$ in (m).

In both cases, axioms and theorems will be referred to through numerals running from '100' on, previous steps in a proof through numerals running from '1' on.

(*d*) When appeal is made to one of D1–D7 to abbreviate a given step of a proof, we shall quote the relevant definition after data (*c*1) or (*c*2).

(*e*) If, in a given proof, a component φ of a step (n) has already been established either as a theorem (say Tm) or as a step (say (m)) of the same proof, we shall use 'm' as an abbreviation for φ in writing down (n). Step (1) of theorem T105, for instance, is short for '$((p \vee q) \supset (q \vee p)) \supset ((\sim p \vee q)) \supset (\sim p \vee (q \vee p)))$'.

8. *Sample theorems*:

T104: $(p \supset q) \supset ((r \supset p) \supset (r \supset q))$. $(103, \sim r/r)$ D2

T105: $p \supset (q \vee p)$.

Proof:

(1): $102 \supset ((\sim p \vee (p \vee q)) \supset (\sim p \vee (q \vee p)))$	
	$(103, p \vee q/p, q \vee p/q, \sim p/r)$
(2): $101 \supset (p \supset (q \vee p))$	(102, 1) D2
(3): $p \supset (q \vee p)$	(101, 2)

T106: $p \supset (q \supset p)$. $(105, \sim q/q)$ D2

T107: $p \supset p$.

Proof:

(1): $100 \supset ((p \supset (p \vee p)) \supset (p \supset p))$	
	$(104, p \vee p/p, p/q, p/r)$
(2): $(p \supset (p \vee p)) \supset (p \supset p)$	(100, 1)
(3): $p \supset (p \vee p)$	$(101, p/q)$
(4): $p \supset p$	(2, 3)

Note: T107 was recorded in chapter one as schema T3a.

T108: $p \vee \sim p$.

Proof:

(1): $107 \supset (p \vee \sim p)$ (102, $\sim p/p$, p/q) D2

(2): $p \vee \sim p$ (107, 1)

Note: T108 was recorded in chapter one as schema T1.

T109: $(p \vee (q \vee r)) \supset (q \vee (p \vee r))$.

Proof:

(1): $r \supset (p \vee r)$ (105, r/p, p/q)

(2): $1 \supset ((q \vee r) \supset (q \vee (p \vee r)))$ (103, r/p, $p \vee r/q$, q/r)

(3): $(q \vee r) \supset (q \vee (p \vee r))$ (1, 2)

(4): $3 \supset ((p \vee (q \vee r)) \supset (p \vee (q \vee (p \vee r))))$ (103, $q \vee r/p$, $q \vee (p \vee r)/q$, p/r)

(5): $(p \vee (q \vee r)) \supset (p \vee (q \vee (p \vee r)))$ (3, 4)

(6): $(p \vee r) \supset (q \vee (p \vee r))$ (105, $p \vee r/p$)

(7): $p \supset (p \vee r)$ (101, r/q)

(8): $6 \supset (7 \supset (p \supset (q \vee (p \vee r))))$ (104, $p \vee r/p$, $q \vee (p \vee r)/q$, p/r)

(9): $7 \supset (p \supset (q \vee (p \vee r)))$ (6, 8)

(10): $p \supset (q \vee (p \vee r))$ (7, 9)

(11): $10 \supset (((q \vee (p \vee r)) \vee p) \supset ((q \vee (p \vee r)) \vee (q \vee (p \vee r))))$ (103, $q \vee (p \vee r)/q$, $q \vee (p \vee r)/r$)

(12): $((q \vee (p \vee r)) \vee p) \supset ((q \vee (p \vee r)) \vee (q \vee (p \vee r)))$ (10, 11)

(13): $((q \vee (p \vee r)) \vee (q \vee (p \vee r))) \supset (q \vee (p \vee r))$ (100, $q \vee (p \vee r)/p$)

(14): $13 \supset (12 \supset (((q \vee (p \vee r)) \vee p) \supset (q \vee (p \vee r))))$ (104, $(q \vee (p \vee r)) \vee (q \vee (p \vee r))/p$, $q \vee (p \vee r)/q$, $(q \vee (p \vee r)) \vee p/r$

(15): $12 \supset (((q \vee (p \vee r)) \vee p) \supset (q \vee (p \vee r)))$ (13, 14)

(16): $((q \vee (p \vee r)) \vee p) \supset (q \vee (p \vee r))$ (12, 15)

(17): $(p \vee (q \vee (p \vee r))) \supset ((q \vee (p \vee r)) \vee p)$ (102, $q \vee (p \vee r)/q$)

(18): $16 \supset (17 \supset ((p \vee (q \vee (p \vee r))) \supset (q \vee (p \vee r))))$
(104, $(q \vee (p \vee r)) \vee p/p$, $q \vee (p \vee r)/q$, $p \vee (q \vee (p \vee r))/r$)
(19): $17 \supset ((p \vee (q \vee (p \vee r))) \supset (q \vee (p \vee r)))$
(16, 18)
(20): $(p \vee (q \vee (p \vee r))) \supset (q \vee (p \vee r))$
(17, 19)
(21): $20 \supset (5 \supset ((p \vee (q \vee r)) \supset (q \vee (p \vee r))))$
(104, $p \vee (q \vee (p \vee r))/p$, $q \vee (p \vee r)/q$, $p \vee (q \vee r)/r$)
(22): $5 \supset ((p \vee (q \vee r)) \supset (q \vee (p \vee r)))$
(20, 21)
(23): $(p \vee (q \vee r)) \supset (q \vee (p \vee r))$ (22, 5)

This last proof is more elaborate than its predecessors; it falls into three parts: steps (1)–(5) lead to '$(p \vee (q \vee r)) \supset (p \vee (q \vee (p \vee r)))$'; steps (6)–(20) lead to '$(p \vee (q \vee (p \vee r))) \supset (q \vee (p \vee r))$'; steps (21)–(23) lead from these two formulae to '$(p \vee (q \vee r)) \supset (q \vee (p \vee r))$'.

T110: $(p \supset (q \supset r)) \supset (q \supset (p \supset r))$. (109, $\sim p/p$, $\sim q/q$) D2
T111: $(p \supset q) \supset ((q \supset r) \supset (p \supset r))$.

Proof:
(1): $104 \supset ((r \supset p) \supset ((p \supset q) \supset (r \supset q)))$
(110, $p \supset q/p$, $r \supset p/q$, $r \supset q/r$)
(2): $(r \supset p) \supset ((p \supset q) \supset (r \supset q))$ (104, 1)
(3): $(p \supset q) \supset ((q \supset r) \supset (p \supset r))$ (2, q/p, r/q, p/r)

With T111 at hand, we may adopt a shortcut in recording proofs. Instead of writing:

(1): $\ulcorner\psi \supset \chi\urcorner$
(2): $\ulcorner\varphi \supset \psi\urcorner$
(3): $\ulcorner(\varphi \supset \psi) \supset ((\psi \supset \chi) \supset (\varphi \supset \chi))\urcorner$
(111)
(4): $\ulcorner(\psi \supset \chi) \supset (\varphi \supset \chi)\urcorner$ (2, 3)
(5): $\ulcorner\varphi \supset \chi\urcorner$ (1, 4)

we shall write:

(1): $\ulcorner\psi \supset \chi\urcorner$
(2): $\ulcorner\varphi \supset \psi\urcorner$
$\supset \chi\urcorner$ (1),

where (2) stands for $\ulcorner\varphi \supset \chi\urcorner$. This convention is applied in T113 and T114.

T112: $(p \vee (q \vee r)) \supset (p \vee (r \vee q))$.

Proof:

(1): $(q \vee r) \supset (r \vee q)$	$(102, q/p, r/q)$
(2): $1 \supset ((p \vee (q \vee r)) \supset (p \vee (r \vee q)))$	$(103, q \vee r/p, r \vee q/q, p/r)$
(3): $(p \vee (q \vee r)) \supset (p \vee (r \vee q))$	$(1, 2)$

With T109 and T112 at hand, we may prove the two conditionals which make up schema T9b of chapter one.

T113: $(p \vee (q \vee r)) \supset ((p \vee q) \vee r)$.

Proof: $(p \vee (q \vee r)) \supset (p \vee (r \vee q))$	(112)
$\supset (r \vee (p \vee q))$	$(109, r/q, q/r)$
$\supset ((p \vee q) \vee r)$	$(102, r/p, p \vee q/q)$

T114: $((p \vee q) \vee r) \supset (p \vee (q \vee r))$.

Proof: $((p \vee q) \vee r) \supset (r \vee (p \vee q))$	$(102, p \vee q/p, r/q)$
$\supset (p \vee (r \vee q))$	$(109, r/p, p/q, q/r)$
$\supset (p \vee (q \vee r))$	$(112, r/q, q/r)$

In view of D4 the conjunction:

$$114 \,.\, 113,$$

abbreviates into '$((p \vee q) \vee r) \equiv (p \vee (q \vee r))$'. To deduce the conjunction:

$$114 \,.\, 113,$$

from T114 and T113, however, we need the familiar rule of Adjunction. We shall derive it below.

28. THE SENTENTIAL CALCULUS, VERSION II

Let us understand by an *instance of a well-formed formula* φ any result of replacing the sentential dummies $\zeta_1, \zeta_2, \ldots, \zeta_n$ $(n \geq 0)$ in φ by the well-formed formulae $\psi_1, \psi_2, \ldots, \psi_m$ $(m \leq n)$.[11] It is clear that all the instances of axioms A100–A103 are provable as theorems through R3a. The following three instances of A101: '$p \supset (p \vee p)$', '$q \supset (q \vee p)$', and '$(r \supset r) \supset ((r \supset r) \vee s)$', for example, are provable as theorems through R3a; the first one is provable through substitution of 'p' for 'q', the

[11]Note that the word 'instance' has a different meaning here than in chapters one and two.

second through substitution of 'q' for 'p' and 'p' for 'q', the third through substitution of '$r \supset r$' for 'p' and 's' for 'q' in A101.

In the present version of the sentential calculus we enlarge our set of axioms to include all formulae of the form:

$$\ulcorner(\varphi \vee \varphi) \supset \varphi\urcorner \quad (1),$$

$$\ulcorner\varphi \supset (\varphi \vee \psi)\urcorner \quad (2),$$

$$\ulcorner(\varphi \vee \psi) \supset (\psi \vee \varphi)\urcorner \quad (3),$$

and

$$\ulcorner(\varphi \supset \psi) \supset ((\chi \vee \varphi) \supset (\chi \vee \psi))\urcorner \quad (4).$$

To stress the distinction between the four formulae A100–A103 and the four infinite sets of formulae (1)–(4), we shall call (1)–(4) *metaaxioms* and number them MA100–MA103.

The shift from A100–A103 to MA100–MA103 has many advantages:

1. All the instances of axioms A100–A103, which in section 27 had to be deduced one by one from A100–A103 through use of R3a, now automatically qualify as axioms and hence as theorems.

2. Because of 1 proofs become simpler to record. The proof of T105, for example, will now read:

T105: $p \supset (q \vee p)$.

Proof:

(1): $((p \vee q) \supset (q \vee p)) \supset ((\sim p \vee (p \vee q)) \supset (\sim p \vee (q \vee p)))$ (MA103)

(2): $(p \supset (p \vee q)) \supset (p \supset (q \vee p))$ (MA102, 1) D2

(3): $p \supset (q \vee p)$ (MA101, 2)

Whereas (1) previously had to be justified as arising from the substitution of '$p \vee q$' for 'p', '$q \vee p$' for 'q', and '$\sim p$' for 'r' in A103, it is now an axiom and hence may figure in the proof of T105 without any appended reference to R3a.

3. Although the instances of axioms A100–A103 automatically qualify as axioms, the instances of theorems T104–T114 do not automatically qualify as theorems; they still have to be deduced from T104–T114 through R3a. Whereas, however, R3a had to be assumed in section 27 as a primitive rule of deduction, it is forthcoming here as a derived one.

In our present version of the sentential calculus we may accordingly drop paragraphs 5 and 6 of section 27 in favor of:

5. *Metaaxioms and rule of deduction*:

(*a*) Metaaxioms:

MA100: $\ulcorner(\varphi \vee \varphi) \supset \varphi\urcorner$;
MA101: $\ulcorner\varphi \supset (\varphi \vee \psi)\urcorner$;

MA102: $\ulcorner(\varphi \vee \psi) \supset (\psi \vee \varphi)\urcorner$;
MA103: $\ulcorner(\varphi \supset \psi) \supset ((\chi \vee \varphi) \supset (\chi \vee \psi))\urcorner$.

(*b*) Rule of deduction:

R1: From φ and $\ulcorner\varphi \supset \psi\urcorner$ one may infer ψ (Detachment).

6. *Definitions of a proof and of a theorem*: A proof is a finite sequence of well-formed formulae each one of which is an axiom or is obtainable from previous formulae in the sequence through Detachment; a theorem is the last formula of a proof.

We next prove R3a as a derived rule of deduction:

R3a: If φ is a theorem, then so is $\varphi\frac{\psi}{\zeta}$, where $\varphi\frac{\psi}{\zeta}$ is the result of replacing at each one of its occurrences in φ the sentential dummy ζ by the well-formed formula ψ (Sentential Substitution).

Proof: Let the proof of φ be given and let the axioms figuring in it be $\chi_1, \chi_2, \ldots, \chi_n$. If we replace each one of $\chi_1, \chi_2, \ldots, \chi_n$ by the axioms $\chi_1\frac{\psi}{\zeta}, \chi_2\frac{\psi}{\zeta}, \ldots, \chi_n\frac{\psi}{\zeta}$ and replace ζ by ψ in every formula obtained from $\chi_1, \chi_2, \ldots, \chi_n$ through R1, we are left with a sequence of axioms and formulae obtainable from axioms through R1. Since the last formula in that sequence is $\varphi\frac{\psi}{\zeta}$, $\varphi\frac{\psi}{\zeta}$ is a theorem if φ is. Q.E.D.[11a]

Let, for example, φ be T105: '$p \supset (q \vee p)$', and let $\varphi\frac{\psi}{\zeta}$ be '$r \supset (q \vee r)$'. The proof of T105 may first be expanded to read:

(1): $(p \vee q) \supset (q \vee p)$	(MA102)
(2): $((p \vee q) \supset (q \vee p)) \supset ((\sim p \vee (p \vee q)) \supset$ $(\sim p \vee (q \vee p)))$	(MA103)
(3): $(p \supset (p \vee q)) \supset (p \supset (q \vee p))$	(1, 2) D2
(4): $p \supset (p \vee q)$	(MA101)
(5): $p \supset (q \vee p)$	(3, 4)

If we replace 'p' by 'r' in the three axioms (1), (2), and (4), and in the two formulae (3) and (5) obtained from (1), (2), and (4) through R1, we get the following proof of '$r \supset (q \vee r)$':

(1): $(r \vee q) \supset (q \vee r)$	(MA102)
(2): $((r \vee q) \supset (q \vee r)) \supset ((\sim r \vee (r \vee q)) \supset (\sim r \vee (q \vee r)))$	(MA103)
(3): $(r \supset (r \vee q)) \supset (r \supset (q \vee r))$	(1, 2) D2

[11a]The proof of R3a is based upon the fact that any instance $\chi_i\frac{\psi}{\zeta}$ of an axiom χ_i of version II is itself an axiom. Note, however, that if a formula like 'p' were appointed, besides the infinite sets of formula MA100–MA103, as an axiom of version II, then R3a would no longer hold.

(4): $r \supset (r \vee q)$ (MA101)
(5): $r \supset (q \vee r)$ (3, 4)

With R1 and R3a at hand we could resume proving individual theorems. We shall prove instead what we called above *metatheorems*, that is, theorems about the sentential calculus. These metatheorems will fall into two main groups:

1. One will consist of derived rules of deduction like R3a above which declare certain formulae of the sentential calculus to be theorems if other formulae of the sentential calculus are theorems.
2. The other will consist of metatheorems like MT104–MT170 below which declare certain formulae of the sentential calculus to be theorems and give instructions for proving them to be theorems. For example, MT105 will declare all the formulae:

$$\ulcorner \varphi \supset (\psi \vee \varphi) \urcorner,$$

to be theorems and give instructions for proving them to be theorems; if $\ulcorner \varphi \supset (\psi \vee \varphi) \urcorner$ is '$p \supset (q \vee p)$', then upon following the instructions of MT105 we shall obtain the first proof displayed in the preceding paragraph; if, on the other hand, $\ulcorner \varphi \supset (\psi \vee \varphi) \urcorner$ is '$r \supset (q \vee r)$', then upon following the instructions of MT105 we shall obtain the second proof displayed in the preceding paragraph.

By switching from theorems to metatheorems, we spare ourselves considerable labor since one metatheorem yields an infinite number of theorems which otherwise would have to be proved one by one as instances of a given theorem; we also spare ourselves the trouble of recording substitutions. For illustration's sake we shall restate T104–T114 as metatheorems and offer proofs for the first five of them; the reader is asked to provide proofs for the remaining six; they can be copied from the proofs of T109–T114.

Note: In the following proofs we shall quote a given metatheorem in full only if it contains Greek letters different from those of its original formulation; otherwise we shall refer to it through its reference numeral; each numeral 'n' from '100' on is now short for 'MAn' or 'MTn'.

MT104: $\ulcorner (\varphi \supset \psi) \supset ((\chi \supset \varphi) \supset (\chi \supset \psi)) \urcorner$. (103) D2
MT105: $\ulcorner \varphi \supset (\psi \vee \varphi) \urcorner$.
Proof:
(1): $\ulcorner 102 \supset ((\sim\varphi \vee (\varphi \vee \psi)) \supset (\sim\varphi \vee (\psi \vee \varphi))) \urcorner$ (103)
(2): $\ulcorner 101 \supset (\varphi \supset (\zeta \vee \varphi)) \urcorner$ (102, 1) D2
(3): $\ulcorner \varphi \supset (\psi \vee \varphi) \urcorner$ (101, 2)

MT105 corresponds to schema T7b of chapter one.

MT106: $\ulcorner\varphi \supset (\psi \supset \varphi)\urcorner$. (105) D2

MT107: $\ulcorner\varphi \supset \varphi\urcorner$.

Proof:

(1):	$\ulcorner 100 \supset ((\varphi \supset (\varphi \vee \varphi)) \supset (\varphi \supset \varphi)\urcorner$	(104)
(2):	$\ulcorner(\varphi \supset (\varphi \vee \varphi)) \supset (\varphi \supset \varphi)\urcorner$	(100, 1)
(3):	$\ulcorner\varphi \supset (\varphi \vee \varphi)\urcorner$	(101)
(4):	$\ulcorner\varphi \supset \varphi\urcorner$	(2, 3)

MT108: $\ulcorner\varphi \vee \sim\varphi\urcorner$.

Proof:

(1):	$\ulcorner 107 \supset (\varphi \vee \sim\varphi)\urcorner$	(102) D2
(2):	$\ulcorner\varphi \vee \sim\varphi\urcorner$	(107, 1)

MT109: $\ulcorner(\varphi \vee (\psi \vee \chi)) \supset (\psi \vee (\varphi \vee \chi))\urcorner$.

MT110: $\ulcorner(\varphi \supset (\psi \supset \chi)) \supset (\psi \supset (\varphi \supset \chi))\urcorner$.

MT111: $\ulcorner(\varphi \supset \psi) \supset ((\psi \supset \chi) \supset (\varphi \supset \psi))\urcorner$.

MT112: $\ulcorner(\varphi \vee (\psi \vee \chi)) \supset (\varphi \vee (\chi \vee \psi))\urcorner$.

MT113: $\ulcorner(\varphi \vee (\psi \vee \chi)) \supset ((\varphi \vee \psi) \vee \chi)\urcorner$.

MT114: $\ulcorner((\varphi \vee \psi) \vee \chi) \supset (\varphi \vee (\psi \vee \chi))\urcorner$.

The next metatheorem will be used in the proof of R4, the rule of Adjunction.

MT115: $\ulcorner\varphi \supset (\psi \supset (\varphi \,.\, \psi))\urcorner$.

Proof:

(1):	$\ulcorner(\sim\varphi \vee \sim\psi) \vee \sim(\sim\varphi \vee \sim\psi)\urcorner$	(108)
(2):	$\ulcorner 1 \supset (\sim\varphi \vee (\sim\psi \vee \sim(\sim\varphi \vee \sim\psi)))\urcorner$	(114)
(3):	$\ulcorner\varphi \supset (\psi \supset (\varphi \,.\, \psi))\urcorner$	(1, 2) D2 twice, D1

R4: If φ and ψ are theorems, then so is $\ulcorner\varphi \,.\, \psi\urcorner$ (Adjunction).[11b]

Proof: From MT115 and the hypothesis that φ is a theorem, it follows by Detachment that $\ulcorner\psi \supset (\varphi \,.\, \psi)\urcorner$ is a theorem; and from $\ulcorner\psi \supset (\varphi \,.\, \psi)\urcorner$ and the hypothesis that ψ is a theorem, it follows by Detachment that $\ulcorner\varphi \,.\, \psi\urcorner$ is a theorem. Q.E.D.

Note: Applications of R4 may be recorded as follows:

(*a*) If a step (n) results from the adjunction of two steps (l) and (m), we shall write:

(n): $l \,.\, m$ (A, l, m).

[11b]Note that the rule of Adjunction bore a different reference numeral in chapters one and two; the same remark will apply below to the rules of Interchange, Universalization, and Relettering.

(*b*) The three steps:

(n): $(l\ .\ m) \supset$ —— . . .
(o): $l\ .\ m$ (A, l, m)
(p): —— (D, o, n),

may be condensed into:

(n): $(l\ .\ m) \supset$ —— . . .
(o): —— ((A, l, m), n).

With R4 at hand, we can strengthen to the biconditional some of the previous metatheorems:

MT116: $\ulcorner(\varphi \vee \varphi) \equiv \varphi\urcorner$.
Proof:
(1): $\ulcorner\varphi \supset (\varphi \vee \varphi)\urcorner$ (101)
(2): $\ulcorner(\varphi \vee \varphi) \equiv \varphi\urcorner$ (A, 100, 1) D4
MT116 corresponds to schema T4b of chapter one.

MT117: $\ulcorner(\varphi \vee \psi) \equiv (\psi \vee \varphi)\urcorner$.
Proof:
(1): $\ulcorner(\psi \vee \varphi) \supset (\varphi \vee \psi)\urcorner$ (102)
(2): $\ulcorner(\varphi \vee \psi) \equiv (\psi \vee \varphi)\urcorner$ (A, 102, 1) D4
MT117 corresponds to schema T8b of chapter one.

MT118: $\ulcorner\varphi \equiv \varphi\urcorner$. (A, 107, 107) D4
MT118 corresponds to schema T9b of chapter one.

MT119: $\ulcorner(\varphi \supset (\psi \supset \chi)) \equiv (\psi \supset (\varphi \supset \chi))\urcorner$.
Proof:
(1): $\ulcorner(\psi \supset (\varphi \supset \chi)) \supset (\varphi \supset (\psi \supset \chi))\urcorner$ (110)
(2): $\ulcorner(\varphi \supset (\psi \supset \chi)) \equiv (\psi \supset (\varphi \supset \chi))\urcorner$ (A, 110, 1) D4

MT120: $\ulcorner((\varphi \vee \psi) \vee \chi) \equiv (\varphi \vee (\psi \vee \chi))\urcorner$. (A, 114, 113) D4
MT120 corresponds to schema T8b of chapter one.

MT121: $\ulcorner(\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))\urcorner$.
Proof:
(1): $\ulcorner((\sim\varphi \vee \sim\varphi) \supset \sim\varphi)\ ((\chi \vee (\sim\varphi \vee \sim\varphi)) \supset (\chi \vee \sim\varphi))\urcorner$ (103)
(2): $\ulcorner(\chi \vee (\sim\varphi \vee \sim\varphi)) \supset (\chi \vee \sim\varphi)\urcorner$ (100, 1)
$\supset (\varphi \supset \chi)\urcorner$ (102) D2
(3): $\ulcorner((\chi \vee \sim\varphi) \vee \sim\varphi) \supset (\chi \vee (\sim\varphi \vee \sim\varphi))\urcorner$ (114)
$\supset (\varphi \supset \chi)\urcorner$ (2)
(4): $\ulcorner(\varphi \supset (\varphi \supset \chi)) \supset (\sim\varphi \vee (\chi \vee \sim\varphi))\urcorner$ (112) D2
$\supset ((\chi \vee \sim\varphi) \vee \sim\varphi)\urcorner$ (102)
$\supset (\varphi \supset \chi)\urcorner$ (3)

(5): $\ulcorner 4 \supset (((\varphi \supset \psi) \supset (\varphi \supset (\varphi \supset \chi))) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi)))\urcorner$ (104)

(6): $\ulcorner((\varphi \supset \psi) \supset (\varphi \supset (\varphi \supset \chi))) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))\urcorner$ (4, 5)

(7): $\ulcorner(\psi \supset (\varphi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset (\varphi \supset \chi)))\urcorner$ (104)

(8): $\ulcorner(\varphi \supset (\psi \supset \chi)) \supset (\psi \supset (\varphi \supset \chi))\urcorner$ (110)
$\supset ((\varphi \supset \psi) \supset (\varphi \supset (\varphi \supset (\varphi \supset \chi)))\urcorner$ (7)
$\supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))\urcorner$ (6)

MT122: $\ulcorner\varphi \supset {\sim}{\sim}\varphi\urcorner$. (108) D2

MT123: $\ulcorner{\sim}{\sim}\varphi \supset \varphi\urcorner$.

Proof:

(1): $\ulcorner{\sim}\varphi \supset {\sim}{\sim}{\sim}\varphi\urcorner$ (122)
(2): $\ulcorner 1 \supset (108 \supset (\varphi \vee {\sim}{\sim}{\sim}\varphi))\urcorner$ (103)
(3): $\ulcorner 108 \supset (\varphi \vee {\sim}{\sim}{\sim}\varphi)\urcorner$ (1, 2)
$\supset ({\sim}{\sim}\varphi \supset \varphi)\urcorner$ (102) D2
(4): $\ulcorner{\sim}{\sim}\varphi \supset \varphi\urcorner$ (108, 3)

MT124: $\ulcorner\varphi \equiv {\sim}{\sim}\varphi\urcorner$. (A, 122, 123) D4

MT124 corresponds to schema T5 of chapter one.

MT125: $\ulcorner(\varphi \supset \psi) \supset ({\sim}\psi \supset {\sim}\varphi)\urcorner$.

Proof:

(1): $\ulcorner(\varphi \supset {\sim}\psi) \supset (\psi \supset {\sim}\varphi)\urcorner$ (102) D2
(2): $\ulcorner(\psi \supset {\sim}{\sim}\psi) \supset ((\varphi \supset \psi) \supset (\varphi \supset {\sim}{\sim}\psi))\urcorner$ (104)
(3): $\ulcorner\psi \supset {\sim}{\sim}\psi\urcorner$ (122)
(4): $\ulcorner(\varphi \supset \psi) \supset (\varphi \supset {\sim}{\sim}\psi)\urcorner$ (2, 3)
$\supset ({\sim}\psi \supset {\sim}\varphi)\urcorner$ (1)

MT126: $\ulcorner(\varphi \, . \, \psi) \supset \varphi\urcorner$.

Proof:

(1): $\ulcorner({\sim}\varphi \vee {\sim}\psi) \supset {\sim}(\varphi \, . \, \psi)\urcorner$ (122) D1
(2): $\ulcorner 1 \supset ((\varphi \vee ({\sim}\varphi \vee {\sim}\psi)) \supset (\varphi \vee {\sim}(\varphi \, . \, \psi)))\urcorner$ (103)
(3): $\ulcorner(\varphi \vee ({\sim}\varphi \vee {\sim}\psi)) \supset (\varphi \vee {\sim}(\varphi \, . \, \psi))\urcorner$ (1, 2)
(4): $\ulcorner 108 \supset ((\varphi \vee {\sim}\varphi) \vee {\sim}\psi))\urcorner$ (101)
$\supset (\varphi \vee ({\sim}\varphi \vee {\sim}\psi))\urcorner$ (114)
$\supset (\varphi \vee {\sim}(\varphi \, . \, \psi))\urcorner$ (3)
$\supset ((\varphi \, . \, \psi) \supset \varphi)\urcorner$ (102) D2
(5): $\ulcorner(\varphi \, . \, \psi) \supset \varphi\urcorner$ (108, 4)

MT127: $\ulcorner(\varphi \,.\, \psi) \supset \psi\urcorner$.

Proof:

(1): $\ulcorner(\sim\psi \vee \sim\varphi) \supset (\sim\varphi \vee \sim\psi)\urcorner$	(102)	
(2): $\ulcorner 1 \supset ((\varphi \,.\, \psi) \supset (\psi \,.\, \varphi))\urcorner$	(125)	D1
(3): $\ulcorner(\varphi \,.\, \psi) \supset (\psi \,.\, \varphi)\urcorner$	(1, 2)	
$\supset \psi\urcorner$	(126)	

MT126–MT127 correspond to schemata T6a–T6b of chapter one.

We may introduce at this point a second type of proofs, technically called *derivations from assumption formulae.*

A derivation from the assumption formula $\varphi_1, \varphi_2, \ldots, \varphi_n$ $(n \geq 0)$ is a finite sequence of well-formed formulae each one of which is an axiom or an assumption formula or is obtainable from previous formulae in the sequence through Detachment; a well-formed formula ψ is said to be *derivable from the assumption formulae* $\varphi_1, \varphi_2, \ldots, \varphi_n$ if it is the last formula of a derivation from the assumption formulae $\varphi_1, \varphi_2, \ldots, \varphi_n$.[12]

We shall use:

$$\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi,$$

as an abbreviation for the phrase 'ψ is derivable from the assumption formulae $\varphi_1, \varphi_2, \ldots, \varphi_n$' or 'There exists a derivation of ψ from the assumption formulae $\varphi_1, \varphi_2, \ldots, \varphi_n$'. It follows from the above definitions that a derivation from the null set of assumption formulae is what we called above a *proof* and that the last formula of a derivation from the null set of assumption formulae is what we called above a *theorem*;

$$\vdash \varphi$$

may therefore be read: 'φ is a theorem'.

The following are two sample derivations from assumption formulae.

Example 1: We are given the two assumption formulae φ and $\ulcorner\varphi \supset \psi\urcorner$, and asked to derive from them the formula $\ulcorner\psi \vee \chi\urcorner$:

(1): φ	
(2): $\ulcorner\varphi \supset \psi\urcorner$	
(3): ψ	(D, 1, 2)
(4): $\ulcorner\psi \supset (\psi \vee \chi)\urcorner$	(101)
(5): $\ulcorner\psi \vee \chi\urcorner$	(D, 3, 4);

hence: $\varphi, \ulcorner\varphi \supset \psi\urcorner \vdash \ulcorner\psi \vee \chi\urcorner$.

[12]The deductions of chapters one and two were essentially derivations from assumption formulae with premises playing the part of assumption formulae and logical truths playing the part of axioms.

Example 2: We are given the three assumption formulae: $\ulcorner(\varphi \,.\, \psi) \supset \chi\urcorner$, φ, and ψ, and asked to derive from them the formula χ:

(1): $(\varphi \,.\, \psi) \supset \chi\urcorner$	
(2): φ	
(3): ψ	
(4): $\ulcorner\varphi \supset (\psi \supset (\varphi \,.\, \psi))\urcorner$	(115)
(5): $\ulcorner\psi \supset (\varphi \,.\, \psi)\urcorner$	(D, 2, 4)
(6): $\ulcorner\varphi \,.\, \psi\urcorner$	(D, 3, 5)
(7): χ	(D, 1, 6);

hence: $\ulcorner(\varphi \,.\, \psi) \supset \chi\urcorner, \varphi, \psi \vdash \chi$.

Note: (4) may be regarded as standing for its own proof and hence as being short for a sequence of axioms and formulae obtained from axioms through R1.

A formula derivable from a non-null set of assumption formulae need not be provable as a theorem; χ above, for instance, need not be provable as a theorem. The interest of a derivation lies in the fact that if a formula ψ is derivable from the assumption formulae $\varphi_1, \varphi_2, \ldots, \varphi_n$, then the conditional:

$$\ulcorner\varphi_1 \supset (\varphi_2 \supset (\ldots \supset (\varphi_n \supset \psi) \ldots))\urcorner,$$

is provable as a theorem. For example, since

$$\varphi, \varphi \supset \psi \vdash \ulcorner\psi \vee \chi\urcorner,$$

then the conditional:

$$\ulcorner\varphi \supset ((\varphi \supset \psi) \supset (\psi \vee \chi))\urcorner,$$

is provable as a theorem. Similarly, since

$$\ulcorner(\varphi \,.\, \psi) \supset \chi\urcorner, \varphi, \psi \vdash \chi,$$

then the conditional:

$$\ulcorner((\varphi \,.\, \psi) \supset \chi) \supset (\varphi \supset (\psi \supset \chi))\urcorner,$$

is provable as a theorem.

The rule allowing us to deduce:

$$\ulcorner\varphi_1 \supset (\varphi_2 \supset (\ldots \supset (\varphi_n \supset \psi) \ldots))\urcorner$$

from

$$\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi,$$

will be called the rule of *Conditionalization* and numbered R5. The following proof may be skipped at first reading.

deduction the

R5: If $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$, then $\ulcorner \varphi_1 \supset (\varphi_2 \supset (\ldots \supset (\varphi_n \supset \psi) \ldots)) \urcorner$ is a theorem (Conditionalization).[13]

Proof: Our proof will consist of two steps:

Step 1: We shall first prove that if

$$\varphi_1, \varphi_2, \ldots, \varphi_{n-1}, \varphi_n \vdash \psi,$$

then

$$\varphi_1, \varphi_2, \ldots, \varphi_{n-1} \vdash \ulcorner \varphi_n \supset \psi \urcorner.$$

By hypothesis a derivation of ψ from the assumption formulae $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}\ \varphi_n$, is given; let this derivation consist of m formulae $\omega_1, \omega_2, \ldots, \omega_m$ ($m \geq n$), where ω_m is ψ. Replace each formula ω_i by $\ulcorner \varphi_n \supset \omega_i \urcorner$; the last formula will be $\ulcorner \varphi_n \supset \psi \urcorner$, as desired. We shall now show that the resulting sequence of formulae: $\ulcorner \varphi_n \supset \omega_1 \urcorner, \ulcorner \varphi_n \supset \omega_2 \urcorner, \ldots, \ulcorner \varphi_n \supset \omega_m \urcorner$, can be so supplemented as to become a derivation of $\ulcorner \varphi_n \supset \omega_m \urcorner$ from the assumption formulae $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$.

Four possibilities have to be considered: ω_i was an axiom; ω_i was an assumption formula φ_j ($j \leq n - 1$); ω_i was φ_n itself; or ω_i resulted through R1 from two previous formulae ω_k and $\omega_k \supset \omega_i \urcorner$.

Case (a): ω_i was an axiom. Insert between $\ulcorner \varphi_n \supset \omega_i \urcorner$ and whatever formula precedes it the following two formulae:

(p) :	$\ulcorner \omega_i \supset (\varphi_n \supset \omega_i) \urcorner$	(106)
$(p + 1)$:	ω_i	(axiom)

and obtain $\ulcorner \varphi_n \supset \omega_i \urcorner$ by Detachment from (p) and $(p + 1)$.

Case (b): ω_i was an assumption formula φ_j ($j \leq n - 1$). Insert between $\ulcorner \varphi_n \supset \varphi_j \urcorner$ and whatever formula precedes it the following two formulae:

(p):	$\ulcorner \varphi_j \supset (\varphi_n \supset \varphi_j) \urcorner$	(106)
$(p + 1)$:	φ_j	(assumption)

and obtain $\ulcorner \varphi_n \supset \varphi_j \urcorner$ by Detachment from (p) and $(p + 1)$.

Case (c): ω_i was φ_n. Insert the proof previously given of $\ulcorner \varphi_n \supset \varphi_n \urcorner$ (MT107).

Case (d): ω_i resulted through R1 from two previous formulae ω_k and $\ulcorner \omega_k \supset \omega_i \urcorner$, which have now become: $\ulcorner \varphi_n \supset \omega_k \urcorner$ and $\ulcorner \varphi_n \supset (\omega_k \supset \omega_i) \urcorner$.

[13]R5 is sometimes called in the literature *the deduction theorem*. In version I of the sentential calculus a derivation from assumption formulae may be defined as a finite sequence of well-formed formulae each one of which is an axiom or an assumption formula or is obtainable from previous formulae in the sequence through R1 or R3a. R5 does not hold for version I; '$\sim p$', for instance, is derivable from 'p' through R3a, whereas '$p \supset \sim p$' is sententially indeterminate. A restricted variant of R5 holds, however, which we cannot state here.

Insert between $\ulcorner\varphi_n \supset \omega_i\urcorner$ and whatever formula precedes it the following two formulae:

(p): $\ulcorner(\varphi_n \supset (\omega_k \supset \omega_i)) \supset ((\varphi_n \supset \omega_k) \supset (\varphi_n \supset \omega_i))\urcorner$ (121)

$(p + 1)$: $\ulcorner(\varphi_n \supset \omega_k) \supset (\varphi_n \supset \omega_i)\urcorner$ (D, p, and $\ulcorner\varphi_n \supset (\omega_k \supset \omega_i)\urcorner$)

and obtain $\ulcorner\varphi_n \supset \omega_i\urcorner$ by Detachment from $(p + 1)$ and $\ulcorner\varphi_n \supset \omega_k\urcorner$. In all four cases we have so supplemented the sequence of formulae: $\ulcorner\varphi_n \supset \omega_1\urcorner$, $\ulcorner\varphi_n \supset \omega_2\urcorner$, . . . , $\ulcorner\varphi_n \supset \omega_m\urcorner$, as to turn that sequence into a derivation of $\ulcorner\varphi_n \supset \omega_m\urcorner$ or $\ulcorner\varphi_n \supset \psi\urcorner$ from the assumption formulae $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$.

Step 2: By n applications of step 1, it follows that if

$$\varphi_1, \varphi_2, \ldots, \varphi_{n-1}, \varphi_n \vdash \psi,$$

then

$$\vdash \ulcorner\varphi_1 \supset (\varphi_2 \supset (\ldots (\varphi_{n-1} \supset (\varphi_n \supset \psi)) \ldots))\urcorner.$$

But we have seen above that $\vdash \varphi$ if and only if φ is a theorem. If

$$\varphi_1, \varphi_2, \ldots, \varphi_{n-1}, \varphi_n \vdash \psi,$$

the conditional:

$$\ulcorner\varphi_1 \supset (\varphi_2 \supset (\ldots \supset (\varphi_{n-1} \supset (\varphi_n \supset \psi)) \ldots))\urcorner.$$

is therefore a theorem. Q.E.D.

To illustrate step 1, we shall turn the above derivation of $\ulcorner\psi \vee \chi\urcorner$ from the two assumption formulae φ and $\ulcorner\varphi \supset \psi\urcorner$ into a derivation of $\ulcorner(\varphi \supset \psi) \supset (\psi \vee \chi)\urcorner$ from the single assumption formula φ:

(1): $\ulcorner\varphi \supset ((\varphi \supset \psi) \supset \varphi)\urcorner$ (106)
(2): φ (assumption)
(3): $\ulcorner(\varphi \supset \psi) \supset \varphi\urcorner$ (D, 1, 2)
(4): $\ulcorner(\varphi \supset \psi) \supset (\varphi \supset \psi)\urcorner$ (107)
(5): $\ulcorner((\varphi \supset \psi) \supset (\varphi \supset \psi)) \supset (((\varphi \supset \psi) \supset \varphi) \supset ((\varphi \supset \psi) \supset \psi))\urcorner$ (106)
(6): $\ulcorner((\varphi \supset \psi) \supset \varphi) \supset ((\varphi \supset \psi) \supset \psi)\urcorner$ (D, 4, 5)
(7): $\ulcorner(\varphi \supset \psi) \supset \psi\urcorner$ (D, 3, 6)
(8): $\ulcorner(\psi \supset (\psi \vee \chi)) \supset ((\varphi \supset \psi) \supset (\psi \supset (\psi \vee \chi)))\urcorner$ (106)
(9): $\ulcorner\psi \supset (\psi \vee \chi)\urcorner$ (101)
(10): $\ulcorner(\varphi \supset \psi) \supset (\psi \supset (\psi \vee \chi))\urcorner$ (D, 8, 9)
(11): $\ulcorner((\varphi \supset \psi) \supset (\psi \supset (\psi \vee \chi))) \supset (((\varphi \supset \psi) \supset \psi) \supset ((\varphi \supset \psi) \supset (\psi \vee \chi)))\urcorner$ (106)
(12): $\ulcorner((\varphi \supset \psi) \supset \psi) \supset ((\varphi \supset \psi) \supset (\psi \vee \chi))\urcorner$ (D, 10, 11)
(13): $\ulcorner(\varphi \supset \psi) \supset (\psi \vee \chi)\urcorner$ (D, 11, 7)

The reader may proceed from here and turn this derivation into a proof of $\ulcorner\varphi \supset ((\varphi \supset \psi) \supset (\psi \vee \chi))\urcorner$.

Note: When a sequence of $m + n$ ($m \geq 1$ and $n \geq 0$) formulae is given, say:

(1):
(2):
.
.
(m):
($m + 1$):
($m + 2$):
.
.
($m + n$):

which constitutes a derivation of ($m + n$) from (1), (2), . . . , (m) as assumption formulae, we shall on the strength of R5 add to the sequence an extra member, namely:

$(m + n + 1)$: $1 \supset (2 \supset \ldots \supset (m \supset m + n) \ldots))$,

and treat the resulting sequence as a proof of ($m + n + 1$). The deduction of ($m + n + 1$) will be justified by the data '(C, 1, 2, . . . , m, $m + n$)' inserted to the right of ($m + n + 1$).

Steps (1)–(10) of the proof of MT128, for instance, constitute a derivation of (10) from (1) as assumption formula; we accordingly add to the sequence (1)–(10) an extra member, namely:

(11): $\ulcorner 1 \supset (\sim\varphi \supset \sim\psi)\urcorner$,

and treat the resulting sequence (1)–(11) as a proof of (11); the deduction of (11) is justified by the data '(C, 1, 10)' inserted to the right of (11).

MT128: $\ulcorner(\varphi \equiv \psi) \supset (\sim\varphi \equiv \sim\psi)\urcorner$.

Proof:

(1):	$\ulcorner(\varphi \supset \psi) . (\psi \supset \varphi)\urcorner$	
(2):	$\ulcorner 1 \supset (\varphi \supset \psi)\urcorner$	(126)
(3):	$\ulcorner\varphi \supset \psi\urcorner$	(1, 2)
(4):	$\ulcorner 1 \supset (\psi \supset \varphi)\urcorner$	(127)
(5):	$\ulcorner\psi \supset \varphi\urcorner$	(1, 4)
(6):	$\ulcorner 5 \supset (\sim\varphi \supset \sim\psi)\urcorner$	(125)
(7):	$\ulcorner\sim\varphi \supset \sim\psi\urcorner$	(5, 6)
(8):	$\ulcorner 3 \supset (\sim\psi \supset \sim\varphi)\urcorner$	(125)
(9):	$\ulcorner\sim\psi \supset \sim\varphi\urcorner$	(3, 8)

(10): $\ulcorner \sim\varphi \equiv \sim\psi \urcorner$ (A, 7, 9) D4
(11): $\ulcorner (\varphi \equiv \psi) \supset (\sim\varphi \equiv \sim\psi) \urcorner$ (C, 1, 10) D4

Note: We allow uses of R4 as in (10) to shorten derivations.

MT129: $\ulcorner ((\varphi \supset \psi) . (\varphi \supset \chi)) \supset (\varphi \supset (\psi . \chi)) \urcorner$.

Proof:

(1): $\ulcorner (\varphi \supset \psi) . (\varphi \supset \chi) \urcorner$
(2): φ
(3): $\ulcorner 1 \supset (\varphi \supset \psi) \urcorner$ (126)
(4): $\ulcorner \varphi \supset \psi \urcorner$ (1, 3)
(5): ψ (2, 4)
(6): $\ulcorner 1 \supset (\varphi \supset \chi) \urcorner$ (127)
(7): $\ulcorner \varphi \supset \chi \urcorner$ (1, 6)
(8): χ (2, 7)
(9): $\ulcorner \psi . \chi \urcorner$ (A, 5, 8)
(10): $\ulcorner ((\varphi \supset \psi) . (\varphi \supset \chi)) \supset (\varphi \supset (\psi . \chi)) \urcorner$ (C, 1, 2, 9)

MT130: $\ulcorner (\varphi \vee \psi) \supset (\sim\varphi \supset \psi) \urcorner$.

Proof:

(1): $\ulcorner 122 \supset ((\psi \vee \varphi) \supset (\psi \vee \sim\sim\varphi)) \urcorner$ (103)
(2): $\ulcorner (\psi \vee \varphi) \supset (\psi \vee \sim\sim\varphi) \urcorner$ (122, 1)
$\supset (\sim\varphi \supset \psi) \urcorner$ (102) D2
(3): $\ulcorner (\varphi \vee \psi) \supset (\psi \vee \varphi) \urcorner$ (102)
$\supset (\sim\varphi \supset \psi) \urcorner$ (2)

MT131: $\ulcorner (\sim\varphi \supset \psi) \supset (\varphi \vee \psi) \urcorner$.

Proof:

(1): $\ulcorner 123 \supset ((\psi \vee \sim\sim\varphi) \supset (\psi \vee \varphi)) \urcorner$ (103)
(2): $\ulcorner (\psi \vee \sim\sim\varphi) \supset (\psi \vee \varphi) \urcorner$ (1, 123)
$\supset (\varphi \vee \psi) \urcorner$ (102)
(3): $\ulcorner (\sim\varphi \supset \psi) \supset (\psi \vee \sim\sim\varphi) \urcorner$ (102) D2
$\supset (\varphi \vee \psi) \urcorner$ (2)

MT132: $\ulcorner (\varphi \vee \psi) \equiv (\sim\varphi \supset \psi) \urcorner$. (A, 130, 131) D4

MT133: $\ulcorner ((\varphi \supset \psi) . (\chi \supset \omega)) \supset ((\varphi \vee \chi) \supset (\varphi \vee \omega)) \urcorner$.

Proof:

(1): $\ulcorner (\varphi \supset \psi) . (\chi \supset \omega) \urcorner$
(2): $\ulcorner \varphi \vee \chi \urcorner$
(3): $\ulcorner 1 \supset (\varphi \supset \psi) \urcorner$ (126)
(4): $\ulcorner \varphi \supset \psi \urcorner$ (1, 3)
(5): $\ulcorner 1 \supset (\chi \supset \omega) \urcorner$ (127)
(6): $\ulcorner \chi \supset \omega \urcorner$ (1, 5)
(7): $\ulcorner 2 \supset (\sim\varphi \supset \chi) \urcorner$ (130)
(8): $\ulcorner \sim\varphi \supset \chi \urcorner$ (2, 7)

(9):	$\ulcorner 4 \supset (\sim\psi \supset \sim\varphi)\urcorner$	(125)
(10):	$\ulcorner\sim\psi \supset \sim\varphi\urcorner$	(4, 9)
	$\supset \chi\urcorner$	(8)
	$\supset \omega\urcorner$	(6)
(11):	$\ulcorner 10 \supset (\psi \vee \omega)\urcorner$	(131)
(12):	$\ulcorner\psi \vee \omega\urcorner$	(10, 11)
(13):	$\ulcorner((\varphi \supset \psi) . (\chi \supset \omega)) \supset ((\varphi \vee \chi) \supset (\psi \vee \omega))\urcorner$	(C, 1, 2, 12)

MT134: $\ulcorner((\varphi \equiv \psi) . (\chi \equiv \omega)) \supset ((\varphi \vee \chi) \equiv (\psi \vee \omega))\urcorner$.

Proof:

(1):	$\ulcorner(\varphi \equiv \psi) . (\chi \equiv \omega)\urcorner$	
(2):	$\ulcorner 1 \supset (\varphi \equiv \psi)\urcorner$	(126)
(3):	$\ulcorner\varphi \equiv \psi\urcorner$	(1, 2)
(4):	$\ulcorner 3 \supset (\varphi \supset \psi)\urcorner$	(126) D4
(5):	$\ulcorner\varphi \supset \psi\urcorner$	(3, 4)

Similarly:

(6):	$\ulcorner\psi \supset \varphi\urcorner$	
(7):	$\ulcorner\chi \supset \omega\urcorner$	
(8):	$\ulcorner\omega \supset \chi\urcorner$	
(9):	$\ulcorner(5 . 7) \supset ((\varphi \vee \chi) \supset (\psi \vee \omega))\urcorner$	(133)
(10):	$\ulcorner(\varphi \vee \chi) \supset (\psi \vee \omega)\urcorner$	((A, 5, 7), 9)
(11):	$\ulcorner(6 . 8) \supset ((\psi \vee \omega) \supset (\varphi \vee \chi))\urcorner$	(133)
(12):	$\ulcorner(\psi \vee \omega) \supset (\varphi \vee \chi)\urcorner$	((A, 6, 8), 11)
(13):	$\ulcorner(\varphi \vee \chi) \equiv (\psi \vee \omega)\urcorner$	(A, 10, 12) D4
(14):	$\ulcorner((\varphi \equiv \psi) . (\chi \equiv \omega)) \supset ((\varphi \vee \chi) \equiv (\psi \vee \omega))\urcorner$	(C, 1, 13)

We now turn to the rule of Interchange, the proof of which may also be omitted at first reading:

R6: If φ and $\ulcorner\psi \equiv \psi'\urcorner$ are theorems and φ' is like φ except for containing ψ' at zero or more places where φ contains ψ, then φ' is a theorem (Interchange).

Proof: Our proof will consist of two steps:

Step 1: We shall first prove that if φ' is like φ except for containing ψ' at zero or more places where φ contains ψ, then

$$\ulcorner(\psi \equiv \psi') \supset (\varphi \equiv \varphi')\urcorner \tag{1}$$

is a theorem.

Consider the formula φ; it will contain, when expanded into primitive notation, a total number n ($n \geq 0$) of negation and alternation signs. We shall first show (case (a)) that (1) is a theorem for formulae φ which do not contain any primitive connective; we shall next show (case (b))

that if (1) is a theorem for formulae φ which contain m or fewer primitive connectives, then (1) is a theorem for formulae φ which contain $m + 1$ primitive connectives. It will follow from (*a*) and (*b*) that (1) is a theorem for formulae φ which contain any number n of primitive connectives.[14] For if, by (*a*), (1) is a theorem for formulae φ which contain zero primitive connective, then, by (*b*), it is a theorem for formulae φ which contain one primitive connective; and if it is a theorem for formulae φ which contain one primitive connective, then, by (*b*) again, it is a theorem for formulae φ which contain two primitive connectives, and so on up to any desired number n of primitive connectives.

Case (*a*): φ does not contain any primitive connective; then, by the definition of a well-formed formula, φ is a sentential dummy ζ and ψ, which occurs in φ, is also ζ. φ' may contain φ' either at zero or at one place.

Subcase (*a*1): If φ' contains ψ' at zero place, then φ' is ζ and (1) becomes:

$$\ulcorner(\zeta \equiv \psi') \supset (\zeta \equiv \zeta)\urcorner.$$

But

$$\ulcorner(\zeta \equiv \zeta) \supset ((\zeta \equiv \psi') \supset (\zeta \equiv \zeta))\urcorner$$

is a theorem (MT106) and

$$\ulcorner\zeta \equiv \zeta\urcorner$$

is a theorem (MT118);

$$\ulcorner(\zeta \equiv \psi') \supset (\zeta \equiv \zeta)\urcorner$$

is therefore a theorem by R1. Q.E.D.

Subcase (*a*2): If φ' contains ψ' at one place, then φ' is ψ' and (1) becomes:

$$\ulcorner(\zeta \equiv \psi') \supset (\zeta \equiv \psi')\urcorner,$$

which is a theorem (MT107) Q.E.D.

Case (*b*): Assume (1) to be a theorem for all formulae φ which contain m or fewer primitive connectives, and let a formula φ be given which contains $m + 1$ primitive connectives. Then, by the definition of a well-formed formula, φ must either be $\ulcorner\sim\chi\urcorner$ or $\ulcorner\chi \vee \omega\urcorner$, where χ and ω contain m or fewer primitive connectives.

Subcase (*b*1): φ is $\ulcorner\sim\chi\urcorner$; then φ' is $\ulcorner\sim\chi'\urcorner$, where χ' is like χ except for containing ψ' at zero or more places where $\ulcorner\sim\chi\urcorner$ contains ψ, and (1) becomes:

$$\ulcorner(\psi \equiv \psi') \supset (\sim\chi \equiv \sim\chi')\urcorner.$$

[14]The principle underlying this argument, namely: "If a predicate is true of 0 (or of 1, as the case may be) and, being true of any natural number j ($j \leq m$), is true of $m + 1$, then it is true of all natural numbers," is the famous principle of *mathematical induction*; it plays a considerable role in metalogic.

Since, by hypothesis,

$$\ulcorner(\psi \equiv \psi') \supset (\chi \equiv \chi')\urcorner$$

is a theorem and

$$\ulcorner((\psi \equiv \psi') \supset (\chi \equiv \chi')) \supset (((\chi \equiv \chi') \supset (\sim\chi \equiv \sim\chi')) \supset ((\psi \equiv \psi') \supset (\sim\chi \equiv \sim\chi')))\urcorner$$

is a theorem (MT111), then by R1

$$\ulcorner((\chi \equiv \chi') \supset (\sim\chi \equiv \sim\chi')) \supset ((\psi \equiv \psi') \supset (\sim\chi \equiv \sim\chi'))\urcorner$$

is also a theorem. But

$$\ulcorner(\chi \equiv \chi') \supset (\sim\chi \equiv \sim\chi')\urcorner$$

is a theorem (MT128);

$$\ulcorner(\psi \equiv \psi') \supset (\sim\chi \equiv \sim\chi')\urcorner$$

is therefore a theorem by R1. Q.E.D.

Subcase (*b*2): φ is $\ulcorner\chi \vee \omega\urcorner$; then φ' is $\ulcorner\chi' \vee \omega'\urcorner$, where χ' and ω' are respectively like χ and ω except for containing ψ' at zero or more places where χ and ω contain ψ, and (1) becomes:

$$\ulcorner(\psi \equiv \psi') \supset ((\chi \vee \omega) \equiv (\chi' \vee \omega'))\urcorner.$$

Since, by hypothesis,

$$\ulcorner(\psi \equiv \psi') \supset (\chi \equiv \chi')\urcorner$$

and

$$\ulcorner(\psi \equiv \psi') \supset (\omega \equiv \omega')\urcorner$$

are theorems, then by R4

$$\ulcorner((\psi \equiv \psi') \supset (\chi \equiv \chi')) \,.\, ((\psi \equiv \psi') \supset (\omega \equiv \omega'))\urcorner$$

is a theorem; since, on the other hand,

$$\ulcorner(((\psi \equiv \psi') \supset (\chi \equiv \chi')) \,.\, ((\psi \equiv \psi') \supset (\omega \equiv \omega'))) \supset ((\psi \equiv \psi') \supset ((\chi \equiv \chi') \,.\, (\omega \equiv \omega')))\urcorner$$

is a theorem (MT129), then by R1

$$\ulcorner(\psi \equiv \psi') \supset ((\chi \equiv \chi') \,.\, (\omega \equiv \omega'))\urcorner$$

is a theorem. But

$$\ulcorner((\psi \equiv \psi') \supset ((\chi \equiv \chi') \,.\, (\omega \equiv \omega'))) \supset ((((\chi \equiv \chi') \,.\, (\omega \equiv \omega')) \supset ((\chi \vee \omega) \equiv (\chi' \vee \omega'))) \supset ((\psi \equiv \psi') \supset ((\chi \vee \omega) \equiv (\chi' \vee \omega'))))\urcorner$$

is a theorem (MT111);

$$\ulcorner(((\chi \equiv \chi') \,.\, (\omega \equiv \omega')) \supset ((\chi \vee \omega) \equiv (\chi' \vee \omega'))) \supset ((\psi \equiv \psi') \supset ((\chi \vee \omega) \equiv (\chi' \vee \omega')))\urcorner$$

is therefore a theorem by R1. But

$$\ulcorner((\chi \equiv \chi') \,.\, (\omega \equiv \omega')) \supset ((\chi \vee \omega) \equiv (\chi' \vee \omega'))\urcorner$$

is in turn a theorem (MT134);

$$\ulcorner(\psi \equiv \psi') \supset ((\chi \vee \omega) \equiv (\chi' \vee \omega'))\urcorner$$

is therefore a theorem by R1. Q.E.D.

Step 2: If $\ulcorner\psi \equiv \psi'\urcorner$ is a theorem and φ' is like φ except for containing ψ' at zero or more places where φ contains ψ, then by step 1 and R1

$$\ulcorner\varphi \equiv \varphi'\urcorner$$

or

$$\ulcorner(\varphi \supset \varphi') \,.\, (\varphi' \supset \varphi)\urcorner$$

is a theorem; but

$$\ulcorner((\varphi \supset \varphi') \,.\, (\varphi' \supset \varphi)) \supset (\varphi \supset \varphi')\urcorner$$

is a theorem (MT126); hence by R1

$$\ulcorner\varphi \supset \varphi'\urcorner$$

is a theorem; but if

$$\ulcorner\varphi \supset \varphi'\urcorner$$

is a theorem and φ, as assumed, is a theorem, then by R1 again φ' is a theorem. Q.E.D.

Note: When a step (n) follows through R6 from two steps (l) and (m), we shall write:

(n): —— (I, l, n)

The proof of MT135 may serve as a sample.

MT135: $\ulcorner\sim(\varphi \,.\, \sim\varphi)\urcorner$.

Proof:

(1): $\ulcorner 107 \equiv \sim\sim(\sim\varphi \vee \varphi)\urcorner$ (124) D2

(2): $\ulcorner\sim\sim(\sim\varphi \vee \varphi)\urcorner$ (I, 1, 107)

(3): $\ulcorner\sim(\varphi \,.\, \sim\varphi)\urcorner$ (I, 2, 124) D1

MT135 corresponds to schema T2 of chapter one.

13 of the above metatheorems correspond to recorded schemata of chapter one; the following metatheorems correspond to the 35 remaining schemata of that chapter. Their proof is left to the reader.

MT136: $\ulcorner(\varphi \,.\, \varphi) \equiv \varphi\urcorner$.
MT137: $\ulcorner(\varphi \,.\, \psi) \equiv (\psi \,.\, \varphi)\urcorner$.
MT138: $\ulcorner(\varphi \equiv \psi) \equiv (\psi \equiv \varphi)\urcorner$.
MT139: $\ulcorner((\varphi \,.\, \psi) \,.\, \chi) \equiv (\varphi \,.\, (\psi \,.\, \chi))\urcorner$.

MT140: $\ulcorner((\varphi \equiv \psi) \equiv \chi) \equiv (\varphi \equiv (\psi \equiv \chi))\urcorner$.
MT141: $\ulcorner(\varphi \,.\, (\psi \vee \chi)) \equiv ((\varphi \,.\, \psi) \vee (\varphi \,.\, \chi))\urcorner$.
MT142: $\ulcorner(\varphi \vee (\psi \,.\, \chi)) \equiv ((\varphi \vee \psi) \,.\, (\varphi \vee \chi))\urcorner$.
MT143: $\ulcorner(\varphi \supset (\psi \,.\, \chi)) \equiv ((\varphi \supset \psi) \,.\, (\varphi \supset \chi))\urcorner$.
MT144: $\ulcorner(\varphi \supset (\psi \vee \chi)) \equiv ((\varphi \supset \psi) \vee (\varphi \supset \chi))\urcorner$.
MT145: $\ulcorner((\varphi \vee \psi) \supset \chi) \equiv ((\varphi \supset \chi) \,.\, (\psi \supset \chi))\urcorner$.
MT146: $\ulcorner((\varphi \,.\, \psi) \supset \chi) \equiv ((\varphi \supset \psi) \vee (\varphi \supset \chi))\urcorner$.
MT147: $\ulcorner((\varphi \supset \psi) \,.\, (\psi \supset \chi)) \supset (\varphi \supset \chi)\urcorner$.
MT148: $\ulcorner((\varphi \equiv \psi) \,.\, (\psi \equiv \chi)) \supset (\varphi \equiv \chi)\urcorner$.
MT149: $\ulcorner(((\varphi \supset \psi) \,.\, (\chi \supset \omega)) \,.\, (\varphi \vee \chi)) \supset (\psi \vee \omega)\urcorner$.
MT150: $\ulcorner((\varphi \,.\, \psi) \supset \chi) \equiv (\varphi \supset (\psi \supset \chi))\urcorner$.
MT151: $\ulcorner((\varphi \,.\, \psi) \supset \chi) \equiv (\psi \supset (\varphi \supset \chi))\urcorner$.
MT152: $\ulcorner(\varphi \supset \psi) \supset ((\chi \,.\, \varphi) \supset (\chi \,.\, \psi))\urcorner$.
MT153: $\ulcorner(\varphi \supset \psi) \equiv ({\sim}\psi \supset {\sim}\varphi)\urcorner$.
MT154: $\ulcorner(\varphi \equiv \psi) \equiv ({\sim}\psi \equiv {\sim}\varphi)\urcorner$.
MT155: $\ulcorner(\varphi \equiv \psi) \equiv ((\varphi \supset \psi) \,.\, (\psi \supset \varphi))\urcorner$.
MT156: $\ulcorner(\varphi \equiv \psi) \supset (\varphi \supset \psi)\urcorner$.
MT157: $\ulcorner(\varphi \supset \psi) \equiv ({\sim}\varphi \vee \psi)\urcorner$.
MT158: $\ulcorner(\varphi \supset \psi) \equiv {\sim}(\varphi \,.\, {\sim}\psi)\urcorner$.
MT159: $\ulcorner(\varphi \not\equiv \psi) \equiv {\sim}(\varphi \equiv \psi)\urcorner$.
MT160: $\ulcorner(\varphi \not\equiv \psi) \equiv ((\varphi \vee \psi) \,.\, {\sim}(\varphi \,.\, \psi))\urcorner$.
MT161: $\ulcorner{\sim}(\varphi \,.\, \psi) \equiv ({\sim}\varphi \vee {\sim}\psi)\urcorner$.
MT162: $\ulcorner{\sim}(\varphi \vee \psi) \equiv ({\sim}\varphi \,.\, {\sim}\psi)\urcorner$.
MT163: $\ulcorner{\sim}(\varphi \equiv \psi) \equiv ({\sim}\varphi \equiv \psi)\urcorner$.
MT164: $\ulcorner{\sim}(\varphi \equiv \psi) \equiv (\varphi \equiv {\sim}\psi)\urcorner$.
MT165: $\ulcorner(\varphi \supset \psi) \equiv (\varphi \equiv (\varphi \,.\, \psi))\urcorner$.
MT166: $\ulcorner(\varphi \supset \psi) \equiv (\psi \equiv (\varphi \vee \psi))\urcorner$.
MT167: $\ulcorner((\varphi \supset \psi) \,.\, \varphi) \supset \psi\urcorner$.
MT168: $\ulcorner((\varphi \supset \psi) \,.\, {\sim}\psi) \supset {\sim}\varphi\urcorner$.
MT169: $\ulcorner((\varphi \vee \psi) \,.\, {\sim}\varphi) \supset \psi\urcorner$.
MT170: $\ulcorner((\varphi \vee \psi) \,.\, {\sim}\psi) \supset \varphi\urcorner$.

29. THE QUANTIFICATIONAL CALCULUS

1. *Primitive signs.* The primitive signs of the quantificational calculus are:

(*a*) the infinite set of sentential dummies: ‘p’, ‘q’, ‘r’, ‘s’, ‘p'’, ‘q'’, ‘r'’, ‘s'’, ‘p''’, ‘q''’, ‘r''’, ‘s''’, and so on;

(*b*) the infinite sets of n-adic predicate dummies: ‘F^n’, ‘G^n’, ‘H^n’, ‘$F^{n\prime}$’, ‘$G^{n\prime}$’, ‘$H^{n\prime}$’, ‘$F^{n\prime\prime}$’, ‘$G^{n\prime\prime}$’, ‘$H^{n\prime\prime}$’, and so on, for every n from 1 on;

(*c*) the infinite set of argument variables: ‘w’, ‘x’, ‘y’, ‘z’, ‘w'’, ‘x'’, ‘y'’, ‘z'’, ‘w''’, ‘x''’, ‘y''’, ‘z''’, and so on;

(*d*) the two connectives '$\sim$' and 'v';
(*e*) the comma ',';
(*f*) the two parentheses '(' and ')'.

Note: The sentential dummies, predicate dummies, and argument variables of the quantificational calculus are listed in (*a*), (*b*), and (*c*) in so-called *alphabetic order*.

2. *Definition of a formula*: A formula is a finite sequence of primitive signs.

3. *Definition of a well-formed formula*:
(*a*) A sentential dummy is a well-formed formula;
(*b*) $\ulcorner \kappa(\alpha_1, \alpha_2, \ldots, \alpha_n) \urcorner$ is a well-formed formula, where κ is a predicate dummy bearing superscript n ($n \geq 1$) and α_1, α_2, . . . , α_n are argument variables;
(*c*) If φ is a well-formed formula, so is $\ulcorner(\alpha)\varphi\urcorner$, where α is an argument variable;
(*d*) If φ is a well-formed formula, so is $\ulcorner \sim\varphi \urcorner$;
(*e*) If φ and ψ are well-formed formulae, so is $\ulcorner(\varphi \mathsf{v} \psi)\urcorner$.

Note: The superscripts borne by the predicate dummies of 1(*b*) prevent a letter from serving in a given well-formed formula both as a *p*-adic and as a *q*-adic predicate dummy ($p \neq q$). In practice, however, we shall simply write '*F*', '*G*', and '*H*', and assume that a predicate dummy followed by a one-place argument sequence is monadic, a predicate dummy followed by a two-place argument sequence is dyadic, . . . , and a predicate dummy followed by an *n*-place argument sequence is *n*-adic.

4. *Defined signs.* The following eight definitions are adopted: D1–D7 as on page 116, and

D8: $\ulcorner(\mathrm{E}\alpha)\varphi\urcorner \rightarrow \ulcorner\sim(\alpha)\sim\varphi\urcorner$.

5. *Definition of a scope*: φ is the scope of $\ulcorner(\alpha)\urcorner$ in $\ulcorner(\alpha)\varphi\urcorner$.

6. *Definitions of bondage and freedom*:
(*a*) A given occurrence of the argument variable α is bound if it falls within $\ulcorner(\alpha)\urcorner$ or within the scope of $\ulcorner(\alpha)\urcorner$;
(*b*) A given argument variable of α is bound in a given formula φ if at least one of its occurrences in φ is bound;
(*c*) A given occurrence of the argument variable α in a given formula φ is free if it is not bound in φ;
(*d*) A given argument variable α is free in a given formula φ if it is not bound in φ.

7. *Axioms and rules of deduction.* Two versions of the quantificational calculus may be offered, one with axioms, the other with metaaxioms. The first one is sketched here only in outline.

(*a*) Axioms:
A100: $(p \vee p) \supset p$;
A101: $p \supset (p \vee q)$;
A102: $(p \vee q) \supset (q \vee p)$;
A103: $(p \supset q) \supset ((r \vee p) \supset (r \vee q))$;
A200: $(x)F(x) \supset F(y)$;
A201: $(x)(p \supset F(x)) \supset (p \supset (x)F(x))$.[15]

(*b*) Rules of deduction: Detachment, Universalization, Relettering, and Substitution for sentential dummies, predicate dummies, and free argument variables.

The second version, which we adopt here as our official version, calls for six metaaxioms and two rules of deduction.

(*a*) Metaaxioms:
MA100: $\ulcorner(\varphi \vee \varphi) \supset \varphi\urcorner$;
MA101: $\ulcorner\varphi \supset (\varphi \vee \psi)\urcorner$;
MA102: $\ulcorner(\varphi \vee \psi) \supset (\psi \vee \varphi)\urcorner$;
MA103: $\ulcorner(\varphi \supset \psi) \supset ((\chi \vee \varphi) \supset (\chi \vee \varphi))\urcorner$;
MA200: $\ulcorner(\alpha)\varphi \supset \varphi'\urcorner$, where φ' is like φ except for containing free occurrences of α' wherever φ contains free occurrences of α;
MA201: $\ulcorner(\alpha)(\varphi \supset \psi) \supset (\varphi \supset (\alpha)\psi)\urcorner$, where α is not free in φ.

(*b*) Rules of deduction:
R1: From φ and $\ulcorner\varphi \supset \psi\urcorner$ one may deduce ψ (Detachment).
R2: From φ one may deduce $\ulcorner(\alpha)\varphi\urcorner$ (Universalization).

Note: When $\ulcorner(\alpha)\varphi\urcorner$ is deduced from φ, we shall say that α, if it occurs in φ, *is universalized upon.*

8. *Definitions of a proof and of a theorem*: A proof is a finite sequence of well-formed formulae each one of which is an axiom or is obtainable from previous formulae in the sequence through Detachment or Universalization; a theorem is the last formula of a proof.

Note: Since MA100–MA103 and R1 are included among the metaaxioms and rules of deduction of the quantificational calculus, any theorem of the sentential calculus automatically becomes a theorem of the quantificational calculus.

[15]A200–A201 are an adaptation of Whitehead and Russell's axioms for the quantificational calculus in *Principia Mathematica*; A200 is the Q33 of chapter two.

9. *Conventions of proof recording.* We shall follow the conventions of section 27 as here supplemented:

If a step (n) in a given proof follows through R2 from a previous step (m), we shall write:

(n): —— (U, m).

10. *Definitions of a derivation from assumption formulae and of a formula derivable from assumption formulae*: A derivation from the assumption formulae $\varphi_1, \varphi_2, \ldots, \varphi_n$ $(n \geq 0)$ is a finite sequence of well-formed formulae each one of which is an axiom or an assumption formula or is obtainable from previous formulae in the sequence through Detachment or Universalization; a well-formed formula ψ is derivable from the assumption formulae $\varphi_1, \varphi_2, \ldots, \varphi_n$ $(n \geq 0)$ if it is the last formula of a derivation from the assumption formulae $\varphi_1, \varphi_2, \ldots, \varphi_n$. We shall again use:

$$\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi,$$

as an abbreviation for the phrase 'ψ' is derivable from the assumption formulae $\varphi_1, \varphi_2, \ldots, \varphi_n$' or 'There exists a derivation of ψ from the assumption formula $\varphi_1, \varphi_2, \ldots, \varphi_n$'.

Note: Since MA100–MA103 and R1 are included among the metaaxioms and rules of deduction of the quantificational calculus, any formula derivable in the sentential calculus from a given set of assumption formulae is derivable in the quantificational calculus from the same set of assumption formulae.

11. *Sample metatheorems and derived rules of deduction.*

The following variant of R3a is provable for the quantificational calculus:

R3a: Let $\varphi\frac{\psi}{\zeta}$ be the result of replacing at each one of its occurrences in φ the sentential dummy ζ by the well-formed formula ψ. If φ is a theorem and none of the free argument variables of ψ is bound in φ, then $\varphi\frac{\psi}{\zeta}$ is also a theorem. (Sentential Substitution)

Proof: Let the well-formed formulae $\chi_1, \chi_2, \ldots, \chi_m$ $(m \geq 1)$ be the proof of φ; let $\alpha_1, \alpha_2, \ldots, \alpha_n$ $(n \geq 0)$ be all the free argument variables of ψ which are bound in $\chi_1, \chi_2, \ldots, \chi_{m-1}$; let $\beta_1, \beta_2, \ldots, \beta_n$ be, in alphabetic order, the first n argument variables foreign to $\psi, \chi_1, \chi_2, \ldots, \chi_m$; let χ'_i $(i = 1, 2, \ldots, m)$ be the result of replacing at each one of its occurrences in χ_i the argument variable α_j $(j = 1, 2, \ldots, n)$ by the argument variable β_j; and let $\chi'_i\frac{\psi}{\zeta}$ be the result of replacing at each one of its occur-

rences in χ'_i the sentential dummy ζ by the well-formed formula ψ. If $\chi_1, \chi_2, \ldots, \chi_m$ constitute a proof of χ_m or φ, then $\chi'_1\frac{\psi}{\zeta}, \chi'_2\frac{\psi}{\zeta}, \ldots, \chi'_m\frac{\psi}{\zeta}$ constitute a proof of $\chi'_m\frac{\psi}{\zeta}$; note indeed that if χ_i is an axiom, then $\chi'_i\frac{\psi}{\zeta}$ is also an axiom, that if χ_i is obtainable by R1 from two formulae χ_k and χ_l, then $\chi'_i\frac{\psi}{\zeta}$ is also obtainable by R1 from $\chi'_k\frac{\psi}{\zeta}$ and $\chi'_l\frac{\psi}{\zeta}$, and that if χ_i is obtainable by R2 from a formula χ_k, then $\chi'_i\frac{\psi}{\zeta}$ is also obtainable by R2 from $\chi'_k\frac{\psi}{\zeta}$. But, by hypothesis, none of the free argument variables of ψ is bound in χ_m or φ; hence $\chi'_m\frac{\psi}{\zeta}$ is $\chi_m\frac{\psi}{\zeta}$ or $\varphi\frac{\psi}{\zeta}$, and the proof of $\chi'_m\frac{\psi}{\zeta}$ is a proof of $\varphi\frac{\psi}{\zeta}$. Q.E.D.

The following two rules of Argument Substitution and Predicate Substitution can be established by an analogous reasoning:

R3b: Let $\varphi\frac{\beta}{\alpha}$ be the result of replacing at each one of its occurrences in φ the free argument variable α by an occurrence of the argument variable β. If φ is a theorem and β is not bound in φ, then $\varphi\frac{\beta}{\alpha}$ is also a theorem. (Argument Substitution)

R3c: (*a*) Let φ be a well-formed formula, κ be an n-adic ($n \geq 1$) predicate dummy, $\beta_{1_1}, \ldots, \beta_{n_1}, \beta_{1_2}, \ldots, \beta_{n_2}, \ldots, \beta_{1_j}, \ldots, \beta_{n_j}$ be j ($j \geq 1$) sets of n argument variables (not necessarily distinct within each set nor from one set to another), and $\ulcorner\kappa(\beta_{1_1}, \ldots, \beta_{n_1})\urcorner$, $\ulcorner\kappa(\beta_{1_2}, \ldots, \beta_{n_2})\urcorner, \ldots,$ $\ulcorner\kappa(\beta_{1_j}, \ldots, \beta_{n_j})\urcorner$ exhaust all the occurrences of κ in φ;

(*b*) let $\psi\frac{\beta_{1_i}, \ldots, \beta_{n_i}}{\alpha_1, \ldots, \alpha_n}$ ($i = 1, 2, \ldots, j$), where ψ is a well-formed formula and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are distinct argument variables, be the results of simultaneously replacing at each one of their free occurrences in ψ the argument variables $\alpha_1, \alpha_2, \ldots, \alpha_n$ by occurrences of the argument variables $\beta_{1_i}, \beta_{2_i}, \ldots, \beta_{n_i}$, respectively;[16]

(*c*) let φ' be the result of replacing $\ulcorner\kappa(\beta_{1_1}, \ldots, \beta_{n_1})\urcorner$ in φ by $\psi\frac{\beta_{1_1}, \ldots, \beta_{n_1}}{\alpha_1, \ldots, \alpha_n}$, $\ulcorner\kappa(\beta_{1_2}, \ldots, \beta_{n_2})\urcorner$ by $\psi\frac{\beta_{1_2}, \ldots, \beta_{n_2}}{\alpha_1, \ldots, \alpha_n}, \ldots,$ and $\ulcorner\kappa(\beta_{1_j}, \ldots,$ $\beta_{n_j})\urcorner$ by $\psi\frac{\beta_{1_j}, \ldots, \beta_{n_j}}{\alpha_1, \ldots, \alpha_n}$.

[16]The reader will recognize in $\alpha_1, \alpha_2, \ldots, \alpha_n$ the substitution variables of chapter two, section 19.

If φ is a theorem, no occurrence of $\beta_{1_i}, \beta_{2_i}, \ldots, \beta_{n_i}$ is bound in ψ, nor any occurrence of any free argument variable of ψ other than $\alpha_1, \alpha_2, \ldots, \alpha_n$ is bound in φ, then φ' is also a theorem. (Predicate Substitution)

The rule of Adjunction is also provable for the quantificational calculus:

R4: If φ and ψ are theorems, then so is $\ulcorner\varphi \,.\, \psi\urcorner$ (Adjunction).

Proof: Same as in section 28, with 'φ' and 'ψ' now ranging over quantificational rather than sentential formulae.

The following variant of R5 is provable for the quantificational calculus:

R5: If $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ and no free argument variable of $\varphi_1, \varphi_2, \ldots, \varphi_n$ is universalized upon in the derivation of ψ from $\varphi_1, \varphi_2, \ldots, \varphi_n$, then $\ulcorner\varphi_1 \supset (\varphi_2 \supset (\ldots \supset (\varphi_n \supset \psi) \ldots))\urcorner$ is a theorem (Conditionalization).[17]

Proof: A derivation in the quantificational calculus may differ from a derivation in the sentential calculus in two respects:

(*a*) the occurrence of new axioms through MA200–MA201;

(*b*) the occurrence of formulae obtainable from previous formulae through R2.

(*a*) does not call for any revision of the original proof of R5. In view of (*b*), however, a fifth possibility must be considered:

Case (*e*): ω_i resulted by R2 from a previous formula ω_k and, hence, ω_i was $\ulcorner(\alpha)\omega_k\urcorner$, where α is assumed not to be free in any φ_j $(j = 1, 2, \ldots, n)$; ω_k now becomes $\ulcorner\varphi_n \supset \omega_k\urcorner$. Insert between $\ulcorner\varphi_n \supset (\alpha)\omega_k\urcorner$ and whatever formula precedes it the following two formulae:

(p): $\ulcorner(\alpha)(\varphi_n \supset \omega_k) \supset (\varphi_n \supset (\alpha)\omega_k)\urcorner$ (201 and hypothesis on α)
$(p + 1)$: $\ulcorner(\alpha)(\varphi_n \supset \omega_k)\urcorner$ (U, $\ulcorner\varphi_n \supset \omega_k\urcorner$),

and obtain $\ulcorner\varphi_n \supset (\alpha)\omega_k\urcorner$ by Detachment from (p) and $(p + 1)$.

The rest of the proof remains unchanged except for the three Greek letters 'φ', 'ψ', and 'ω' now ranging over quantificational rather than sentential formulae.

Notes: 1. According to R5 a derivation of a formula ψ from the assumption formulae $\varphi_1, \varphi_2, \ldots, \varphi_n$ may now be followed in a proof by the conditional:

$$\ulcorner\varphi_1 \supset (\varphi_2 \supset (\ldots \supset (\varphi_n \supset \psi)) \ldots)\urcorner,$$

[17]The restriction: 'no free argument variable of $\varphi_1, \varphi_2, \ldots, \varphi_n$ is universalized upon in the derivation of ψ', is clearly needed. We may, for instance, obtain '$(x)F(x)$' from '$F(x)$' through R2; yet '$F(x) \supset (x)F(x)$', being quantificationally indeterminate, is clearly not wanted as a theorem.

if none of the free argument variables of $\varphi_1, \varphi_2, \ldots, \varphi_n$ has been universalized upon in the derivation in question. The first eight lines of the proof of MT204, for instance, constitute a derivation of $\ulcorner(\alpha)\psi\urcorner$ from the two assumption formulae $\ulcorner(\alpha)(\varphi \supset \psi)\urcorner$ and $\ulcorner(\alpha)\varphi\urcorner$. The argument variable α is universalized upon in the last line of the derivation, but α happens to be bound in $\ulcorner(\alpha)(\varphi \supset \psi)\urcorner$ and $\ulcorner(\alpha)\varphi\urcorner$. Hence derivation (1)–(8) may be followed in the proof of MT204 by the conditional:

$$\ulcorner(\alpha)(\varphi \supset \psi) \supset ((\alpha)\varphi \supset (\alpha)\psi)\urcorner.$$

2. As we allowed sentential theorems to figure in a sentential derivation, so we shall allow quantificational theorems to figure in a quantificational derivation. Note, however, that if a quantificational theorem χ is to figure in the derivation of a formula ψ from the assumption formulae $\varphi_1, \varphi_2, \ldots, \varphi_n$, then none of the free argument variables of $\varphi_1, \varphi_2, \ldots, \varphi_n$ may be universalized upon in the proof of χ. The proof of MT206, for instance, opens with an eight line derivation in which figure two occurrences of MT204 (lines (4) and (6)) and one occurrence of MT205 (line (2)). One argument variable, namely α, is universalized upon in the proofs of MT204 and MT205, but α is not free in the assumption formula $\ulcorner(\alpha)((\varphi \supset \psi) \,.\, (\psi \supset \varphi))\urcorner$ of derivation (1)–(8). The derivation in question may accordingly be followed in the proof of MT206 by the conditional:

$$\ulcorner(\alpha)(\varphi \equiv \psi) \supset ((\alpha)\varphi \equiv (\alpha)\psi)\urcorner.$$

In the following proofs we shall leave it to the reader to ascertain that none of the argument variables universalized upon in the derivation of a formula ψ from the assumption formulae $\varphi_1, \varphi_2, \ldots, \varphi_n$ and in the proof of any theorem χ which may figure in the derivation of ψ from $\varphi_1, \varphi_2, \ldots, \varphi_n$, is free in $\varphi_1, \varphi_2, \ldots, \varphi_n$.

Of the next metatheorems, MT203 and MT206 will be used in the proof of R6, the rule of Interchange.

MT202: $\ulcorner(\alpha)\varphi \supset \varphi\urcorner$. (200, where α' is α)

MT203: $\ulcorner(\alpha_1)(\alpha_2) \ldots (\alpha_n)\varphi \supset \varphi\urcorner$.

Proof:

(1): $\ulcorner(\alpha_1)(\alpha_2) \ldots (\alpha_n)\varphi \supset (\alpha_2) \ldots (\alpha_n)\varphi\urcorner$ (202)

$\vdots$

$\supset (\alpha_n)\varphi\urcorner$ (202)

$\supset \varphi\urcorner$ (202)

MT204: $\ulcorner(\alpha)(\varphi \supset \psi) \supset ((\alpha)\varphi \supset (\alpha)\psi)\urcorner$.

Proof:

(1): $\ulcorner(\alpha)(\varphi \supset \psi)\urcorner$

(2): $\ulcorner(\alpha)\varphi\urcorner$

(3):	$\ulcorner 1 \supset (\varphi \supset \psi)\urcorner$	(202)
(4):	$\ulcorner \varphi \supset \psi \urcorner$	(1, 3)
(5):	$\ulcorner 2 \supset \varphi \urcorner$	(202)
(6):	φ	(2, 5)
(7):	ψ	(4, 6)
(8):	$\ulcorner (\alpha)\psi \urcorner$	(U, 7)
(9):	$\ulcorner (\alpha)(\varphi \supset \psi) \supset ((\alpha)\varphi \supset (\alpha)\psi)\urcorner$	(C, 1, 2, 8)

MT204 corresponds to Q36d of chapter two.

MT205: $\ulcorner (\alpha)(\varphi \,.\, \psi) \equiv ((\alpha)\varphi \,.\, (\alpha)\psi \urcorner)$.

Proof:

(1):	$\ulcorner (\alpha)(\varphi \,.\, \psi)\urcorner$	
(2):	$\ulcorner 1 \supset (\varphi \,.\, \psi)\urcorner$	(202)
	$\supset \varphi \urcorner$	(126)
(3):	φ	(1, 2)
(4):	$\ulcorner (\alpha)\varphi \urcorner$	(U, 3)
(5):	$\ulcorner 1 \supset (\varphi \,.\, \psi)\urcorner$	(202)
	$\supset \psi \urcorner$	(127)
(6):	ψ	(1, 5)
(7):	$\ulcorner (\alpha)\psi \urcorner$	(U, 6)
(8):	$\ulcorner (\alpha)\varphi \,.\, (\alpha)\psi \urcorner$	(A, 4, 7)
(9):	$\ulcorner (\alpha)(\varphi \,.\, \psi) \supset ((\alpha)\varphi \,.\, (\alpha)\psi)\urcorner$	(C, 1, 8)
(10):	$\ulcorner (\alpha)\varphi \,.\, (\alpha)\psi \urcorner$	
(11):	$\ulcorner 10 \supset (\alpha)\varphi \urcorner$	(126)
	$\supset \varphi \urcorner$	(202)
(12):	φ	(10, 11)
(13):	$\ulcorner 10 \supset (\alpha)\psi \urcorner$	(127)
	$\supset \psi \urcorner$	(202)
(14):	ψ	(10, 13)
(15):	$\ulcorner \varphi \,.\, \psi \urcorner$	(A, 12, 14)
(16):	$\ulcorner (\alpha)(\varphi \,.\, \psi)\urcorner$	(U, 15)
(17):	$\ulcorner ((\alpha)\varphi \,.\, (\alpha)\psi) \supset (\alpha)(\varphi \,.\, \psi)\urcorner$	(C, 10, 16)
(18):	$\ulcorner (\alpha)(\varphi \,.\, \psi) \equiv ((\alpha)\varphi \,.\, (\alpha)\psi)\urcorner$	(A, 9, 17) D4

MT205 corresponds to Q36a of chapter two.

MT206: $\ulcorner (\alpha)(\varphi \equiv \psi) \supset ((\alpha)\varphi \equiv (\alpha)\psi))\urcorner$.

Proof:

(1):	$\ulcorner (\alpha)((\varphi \supset \psi) \,.\, (\psi \supset \varphi))\urcorner$	
(2):	$\ulcorner 1 \equiv ((\alpha)(\varphi \supset \psi) \,.\, (\alpha)(\psi \supset \varphi))\urcorner$	(205)
(3):	$\ulcorner 2 \supset (1 \supset ((\alpha)(\varphi \supset \psi) \,.\, (\alpha)(\psi \supset \varphi)))\urcorner$	(156)
(4):	$\ulcorner 1 \supset ((\alpha)(\varphi \supset \psi) \,.\, (\alpha)(\psi \supset \varphi))\urcorner$	(2, 3)
	$\supset (\alpha)(\varphi \supset \psi)\urcorner$	(126)
	$\supset ((\alpha)\varphi \supset (\alpha)\psi)\urcorner$	(204)
(5):	$\ulcorner (\alpha)\varphi \supset (\alpha)\psi \urcorner$	(1, 4)

(6): $\ulcorner 1 \supset ((\alpha)(\varphi \supset \psi) \,.\, (\alpha)(\psi \supset \varphi))\urcorner$	(2, 3)	
$\supset (\alpha)(\psi \supset \varphi)\urcorner$	(127)	
$\supset ((\alpha)\psi \supset (\alpha)\varphi)\urcorner$	(204)	
(7): $\ulcorner(\alpha)\psi \supset (\alpha)\varphi\urcorner$	(1, 6)	
(8): $\ulcorner(\alpha)\varphi \equiv (\alpha)\psi\urcorner$	(A, 5, 7)	D4
(9): $\ulcorner(\alpha)(\varphi \equiv \psi) \supset ((\alpha)\varphi \equiv (\alpha)\psi)\urcorner$	(C, 1, 8)	D4

MT206 corresponds to Q36e of chapter two.

R6: If φ and $\ulcorner\psi \equiv \psi'\urcorner$ are theorems and φ' is like φ except for containing ψ' at zero or more places where φ contains ψ, then φ' is a theorem (Interchange).

Proof: Our proof will again consist of two steps:

Step 1: We shall first prove that if φ' is like φ except for containing ψ' at zero or more places where φ contains ψ, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all the free argument variables of ψ and ψ', then

$$\ulcorner(\alpha_1)(\alpha_2) \ldots (\alpha_n)(\psi \equiv \psi') \supset (\varphi \equiv \varphi')\urcorner \tag{1}$$

is a theorem.[18]

To establish step 1, we shall consider this time the number of negation signs, alternation signs, and universal quantifiers which occur in φ, when φ is expanded into primitive notation.

Case (a): φ does not contain any negation sign, alternation sign, or universal quantifier; then, by the definition of a well-formed formula, φ is either a sentential dummy ζ or $\ulcorner\kappa(\beta_1, \beta_2, \ldots, \beta_m)\urcorner$, where κ is a predicate dummy and $\beta_1, \beta_2, \ldots, \beta_m$ $(m > 0)$ are argument variables.

Subcase $(a1)$: If φ is a sentential dummy ζ, then ψ, which occurs in φ, is also ζ. φ' may contain ψ' either at zero or at one place. In the first case, (1) becomes

$$\ulcorner(\alpha_1)(\alpha_2) \ldots (\alpha_n)(\zeta \equiv \psi') \supset (\zeta \equiv \zeta)\urcorner;$$

in the second, (1) becomes

$$\ulcorner(\alpha_1)(\alpha_2) \ldots (\alpha_n)(\zeta \equiv \psi') \supset (\zeta \equiv \psi')\urcorner.$$

The first conditional is a theorem by MT106, MT118, and R1, the second by MT203. Q.E.D.

[18]The restriction: '$\alpha_1, \alpha_2, \ldots, \alpha_n$ are all the free argument variables of φ and φ'', is clearly needed.

$$(x = x \equiv x < 1) \supset ((x)(x = x) \equiv (x)(x < 1)),$$

for instance, yields falsehoods like:

$$(0 = 0 \equiv 0 < 1) \supset ((x)(x = x) \equiv (x)(x < 1)).$$

Note, however, that if $\ulcorner\varphi \equiv \varphi'\urcorner$ is a theorem, then $\ulcorner(\alpha_1)(\alpha_2) \ldots (\alpha_n)(\varphi \equiv \varphi')\urcorner$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all the free argument variables of φ and φ', is also a theorem.

Subcase ($a2$): If φ is $\ulcorner\kappa(\beta_1, \beta_2, \ldots, \beta_m)\urcorner$, then ψ, which occurs in φ, is also $\ulcorner\kappa(\beta_1, \beta_2, \ldots, \beta_m)\urcorner$. φ' may contain ψ' either at zero or at one place. In the first case (1) becomes

$$\ulcorner(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\kappa(\beta_1, \beta_2, \ldots, \beta_m) \equiv \psi) \supset (\kappa(\beta_1, \beta_2, \ldots, \beta_m) \equiv \kappa(\beta_1, \beta_2, \ldots, \beta_m))\urcorner,$$

which is a theorem by MT106, MT118, and R1; in the second, (1) becomes

$$\ulcorner(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\kappa(\beta_1, \beta_2, \ldots, \beta_m) \equiv \psi) \supset (\kappa(\beta_1, \beta_2, \ldots, \beta_m) \equiv \psi)\urcorner,$$

which is a theorem by MT203. Q.E.D.

Case (b): Assume (1) to be a theorem for all formulae φ which contain m or fewer negation signs, alternation signs, and universal quantifiers, and let a formula φ be given which contains $m + 1$ negation signs, alternation signs, and universal quantifiers. Then, by the definition of a well-formed formula, φ must be $\ulcorner\sim\chi\urcorner$, or $\ulcorner\chi \vee \omega\urcorner$, or $\ulcorner(\beta)\chi\urcorner$, where χ and ω contain m or fewer negation signs, alternation signs, and universal quantifiers.

Subcase ($b1$): φ is $\ulcorner\sim\chi\urcorner$; then φ' is $\ulcorner\sim\chi'\urcorner$, where χ' is like χ except for containing ψ' at zero or more places where χ contains ψ, and (1) becomes:

$$\ulcorner(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\psi \equiv \psi') \supset (\sim\chi \equiv \sim\chi')\urcorner.$$

By the same argument as in subcase ($b1$), pages 135–136,

$$\ulcorner(\psi \equiv \psi') \supset (\sim\chi \equiv \sim\chi')\urcorner$$

can be shown to be a theorem; but

$$\ulcorner(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\psi \equiv \psi') \supset (\psi \equiv \psi')\urcorner$$

is a theorem (MT203); hence by MT111 and two applications R1

$$\ulcorner(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\psi \equiv \psi') \supset (\sim\chi \equiv \sim\chi')\urcorner$$

is a theorem. Q.E.D.

Subcase ($b2$): φ is $\ulcorner\chi \vee \omega\urcorner$; then φ' is $\ulcorner\chi' \vee \omega'\urcorner$, where χ' and ω' are respectively like χ and ω except for containing ψ' at zero or more places where χ and ω contain ψ, and (1) becomes:

$$\ulcorner(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\psi \equiv \psi') \supset ((\chi \vee \omega) \equiv (\chi' \vee \omega'))\urcorner.$$

By the same argument as in subcase ($b2$), pages 136–137,

$$\ulcorner(\psi \equiv \psi') \supset ((\chi \vee \omega) \equiv (\chi' \vee \omega'))\urcorner$$

can be shown to be a theorem; but

$$\ulcorner(\alpha_1)(\alpha_2)\ldots(\alpha_n)(\psi \equiv \psi') \supset (\psi \equiv \psi')\urcorner$$

is a theorem (MT203); hence by MT111 and two applications of R1

$$\ulcorner(\alpha_1)(\alpha_2) \ldots (\alpha_n)(\psi \equiv \psi') \supset ((\chi \vee \omega) \equiv (\chi' \vee \omega'))\urcorner$$

is a theorem. Q.E.D.

Subcase ($b3$): φ is $\ulcorner(\beta)\chi\urcorner$; then φ' is $\ulcorner(\beta)\chi'\urcorner$, where χ' is like χ except for containing ψ' at zero or more places where χ contains ψ, and (1) becomes

$$\ulcorner(\alpha_1)(\alpha_2) \ldots (\alpha_n)(\psi \equiv \psi') \supset ((\beta)\chi \equiv (\beta)\chi'\urcorner.$$

By hypothesis

$$\ulcorner(\alpha_1)(\alpha_2) \ldots (\alpha_n)(\psi \equiv \psi') \supset (\chi \equiv \chi')\urcorner$$

is a theorem and, hence, by R2

$$\ulcorner(\beta)((\alpha_1)(\alpha_2) \ldots (\alpha_n)(\psi \equiv \psi') \supset (\chi \equiv \chi'))\urcorner$$

is a theorem; but

$$\ulcorner(\beta)((\alpha_1)(\alpha_2) \ldots (\alpha_n)(\psi \equiv \psi') \supset (\chi \equiv \chi')) \supset ((\alpha_1)(\alpha_2) \ldots (\alpha_n)(\psi \equiv \psi') \supset (\beta)(\chi \equiv \chi'))\urcorner$$

is also a theorem (MT201 and hypothesis on $\alpha_1, \alpha_2, \ldots, \alpha_n$);

$$\ulcorner(\alpha_1)(\alpha_2) \ldots (\alpha_n)(\psi \equiv \psi') \supset (\beta)(\chi \equiv \chi')\urcorner$$

is therefore a theorem by R1.

$$\ulcorner(\beta)(\chi \equiv \chi') \supset ((\beta)\chi \equiv (\beta)\chi')\urcorner,$$

on the other hand, is a theorem (MT206); by MT111 and two applications of R1

$$\ulcorner(\alpha_1)(\alpha_2) \ldots (\alpha_n)(\psi \equiv \psi') \supset ((\beta)\chi \equiv (\beta)\chi')\urcorner$$

is therefore a theorem. Q.E.D.

Step 2: If, as assumed, $\ulcorner\psi \equiv \psi'\urcorner$ is a theorem, then by n applications of R2 $\ulcorner(\alpha_1)(\alpha_2) \ldots (\alpha_n)(\psi \equiv \psi')\urcorner$ is also a theorem. But if $\ulcorner(\alpha_1)(\alpha_2) \ldots (\alpha_n)(\psi \equiv \psi')\urcorner$ is a theorem, then by step 1 and R1 $\ulcorner\varphi \equiv \varphi'\urcorner$ is also a theorem; if, finally, $\ulcorner\varphi \equiv \varphi'\urcorner$ is a theorem and if, as assumed, φ is a theorem, then by familiar steps φ' is also a theorem.

Since $\ulcorner(\psi \equiv \psi') \equiv (\psi' \equiv \psi)\urcorner$ is a theorem (MT138), we shall also use the following version of R6:

R6′: If φ and $\ulcorner\psi' \equiv \psi\urcorner$ are theorems and φ' is like φ except for containing ψ' at zero or more places where φ contains ψ, then φ is a theorem (Interchange).

When recording proofs, we shall not discriminate between R6 and R6′.

We finally derive the rule of Relettering:

R7: Let $\psi\frac{\beta}{\alpha}$ be the result of replacing at each one of its free occurrences in ψ the argument variable α by the argument variable β and let φ' be like φ except for containing $\ulcorner(\beta)\psi\frac{\beta}{\alpha}\urcorner$ at a place where φ contains $\ulcorner(\alpha)\psi\urcorner$. If φ is a theorem and β does not occur in ψ, then φ' is a theorem (Relettering).[19]

Proof: If, on one hand, $\psi\frac{\beta}{\alpha}$ is the result of replacing α by β at each one of its free occurrences in ψ and β does not occur in ψ, then

$$\ulcorner(\alpha)\psi \supset \psi\frac{\beta}{\alpha}\urcorner$$

is a theorem (MT200) and, hence, by R2

$$\ulcorner(\beta)\left((\alpha)\psi \supset \psi\frac{\beta}{\alpha}\right)\urcorner$$

is a theorem. If, on the other hand, β does not occur in ψ, then

$$\ulcorner(\beta)\left((\alpha)\psi \supset \psi\frac{\beta}{\alpha}\right) \supset \left((\alpha)\psi \supset (\beta)\psi\frac{\beta}{\alpha}\right)\urcorner$$

is a theorem (MT201) and, hence, by R1

$$\ulcorner(\alpha)\psi \supset (\beta)\psi\frac{\beta}{\alpha}\urcorner$$

is a theorem. Similarly, by MT200, R2, and MT201,

$$\ulcorner(\beta)\psi\frac{\beta}{\alpha} \supset \psi\urcorner,$$

$$\ulcorner(\alpha)\left((\beta)\psi\frac{\beta}{\alpha} \supset \psi\right)\urcorner,$$

and

$$\ulcorner(\alpha)\left((\beta)\psi\frac{\beta}{\alpha} \supset \psi\right) \supset \left((\beta)\psi\frac{\beta}{\alpha} \supset (\alpha)\psi\right)\urcorner$$

are theorems and, hence, by R1,

$$\ulcorner(\beta)\psi\frac{\beta}{\alpha} \supset (\alpha)\psi\urcorner$$

[19]The reader will recognize in φ' what we called above (chapter two, section 19) *a relettering of* φ. Note, however, the weaker restriction placed upon β, which may occur in φ outside ψ.

is a theorem. By R4 and D4

$$\ulcorner(\alpha)\psi \equiv (\beta)\psi\frac{\beta}{\alpha}\urcorner$$

is therefore a theorem. But φ is assumed to be a theorem and φ' to be like φ except for containing $\ulcorner(\beta)\psi\frac{\beta}{\alpha}\urcorner$ at a place where φ contains $\ulcorner(\alpha)\psi\urcorner$; by R6 φ' is therefore a theorem. Q.E.D.

For example, let φ be '$(x)F(x) \supset (\mathrm{E}x)F(x)$' or '$\sim(x)F(x) \vee \sim(x)\sim F(x)$', let ψ be '$F(x)$', let α be 'x', and let β be 'y'; then $\psi\frac{\beta}{\alpha}$ is '$F(y)$' and '$(\beta)\psi\frac{\beta}{\alpha}$' is '$(y)F(y)$'. Since '$(x)F(x) \equiv (y)F(y)$' is a theorem, then '$\sim(y)F(y) \vee \sim(x)\sim F(x)$' is a theorem if '$\sim(x)F(x) \vee \sim(x)\sim F(x)$' itself is a theorem.

Note: If, in a given proof, a step (n) follows from a previous step (m) through R7, we shall write:

(n): —— (R, m).

Four of the above metatheorems correspond to schemata of section 17; the following metatheorems correspond to the 38 remaining schemata of that section. A few proofs are appended; the others are left to the reader.

MT207: $\ulcorner\sim(\alpha)\varphi \equiv (\mathrm{E}\alpha)\sim\varphi\urcorner$.
MT208: $\ulcorner\sim(\mathrm{E}\alpha)\varphi \equiv (\alpha)\sim\varphi\urcorner$.
MT209: $\ulcorner(\alpha)\varphi \equiv \sim(\mathrm{E}\alpha)\sim\varphi\urcorner$.
MT210: $\ulcorner(\mathrm{E}\alpha)\varphi \equiv \sim(\alpha)\sim\varphi\urcorner$. (118) D8
MT211: $\ulcorner(\alpha)(\varphi \supset \psi) \equiv \sim(\mathrm{E}\alpha)(\varphi \,.\, \sim\psi)\urcorner$.

Proof:

(1):	$\ulcorner(\alpha)(\varphi \supset \psi) \equiv (\alpha)(\varphi \supset \psi)\urcorner$	(118)
(2):	$\ulcorner(\alpha)(\varphi \supset \psi) \equiv (\alpha)\sim(\varphi \,.\, \sim\psi)\urcorner$	(I, 1, 158)
(3):	$\ulcorner\sim(\mathrm{E}\alpha)(\varphi \,.\, \sim\psi) \equiv (\alpha)\sim(\varphi \,.\, \sim\psi)\urcorner$	(208)
(4):	$\ulcorner(\alpha)(\varphi \supset \psi) \equiv \sim(\mathrm{E}\alpha)(\varphi \,.\, \sim\psi)\urcorner$	(I, 2, 3)

MT212: $\ulcorner(\alpha)(\varphi \supset \sim\psi) \equiv \sim(\mathrm{E}\alpha)(\varphi \,.\, \psi)\urcorner$.
MT213: $\ulcorner(\mathrm{E}\alpha)(\varphi \,.\, \psi) \equiv \sim(\alpha)(\varphi \supset \sim\psi)\urcorner$.
MT214: $\ulcorner(\mathrm{E}\alpha)(\varphi \,.\, \sim\psi) \equiv \sim(\alpha)(\varphi \supset \psi)\urcorner$.
MT215: $\ulcorner((\alpha)(\psi \supset \chi) \,.\, (\alpha)(\varphi \supset \psi)) \supset (\alpha)(\varphi \supset \chi)\urcorner$.

Proof:

(1):	$\ulcorner(\alpha)(\psi \supset \chi) \,.\, (\alpha)(\varphi \supset \psi)\urcorner$	
(2):	$\ulcorner 1 \supset (\alpha)(\psi \supset \chi)\urcorner$	(126)
	$\supset (\psi \supset \chi)\urcorner$	(202)
(3):	$\ulcorner\psi \supset \chi\urcorner$	(1, 2)

(4):	$\ulcorner 1 \supset (\alpha)(\varphi \supset \psi)$ $\supset (\varphi \supset \psi)\urcorner$	(127) (202)
(5):	$\ulcorner \varphi \supset \psi \urcorner$	(1, 4)
(6):	$\ulcorner (5\,.\,3) \supset (\varphi \supset \chi)\urcorner$	(147)
(7):	$\ulcorner \varphi \supset \chi \urcorner$	((A, 5, 3), 6)
(8):	$\ulcorner (\alpha)(\varphi \supset \chi)\urcorner$	(U, 7)
(9):	$\ulcorner ((\alpha)(\psi \supset \chi)\,.\,(\alpha)(\varphi \supset \psi)) \supset (\alpha)(\varphi \supset \chi)\urcorner$	(C, 1, 8)

MT216: $\ulcorner ((\alpha)(\psi \supset \chi)\,.\,(\mathrm{E}\alpha)(\varphi\,.\,\psi)) \supset (\mathrm{E}\alpha)(\varphi\,.\,\chi)\urcorner$.

MT217: $\ulcorner \varphi' \supset (\mathrm{E}\alpha)\varphi \urcorner$, where φ' is like φ except for containing free occurrences of α' wherever φ contains free occurrences of α.

Proof:

(1):	$\ulcorner (\alpha){\sim}\varphi \supset {\sim}\varphi' \urcorner$	(200 and hypothesis)	
(2):	$\ulcorner 1 \supset (\varphi' \supset {\sim}(\alpha){\sim}\varphi)\urcorner$	(102)	D2
(3):	$\ulcorner \varphi' \supset (\mathrm{E}\alpha)\varphi \urcorner$	(1, 2)	D8

MT218: $\ulcorner (\alpha)\varphi \supset (\mathrm{E}\alpha)\varphi \urcorner$.

MT219: $\ulcorner ((\alpha)(\varphi \supset \psi)\,.\,(\mathrm{E}\alpha)\varphi) \supset (\mathrm{E}\alpha)(\varphi\,.\,\psi)\urcorner$.

Proof:

(1):	$\ulcorner ((\alpha)(\varphi \supset \psi)\,.\,(\mathrm{E}\alpha)(\varphi\,.\,\varphi)) \supset (\mathrm{E}\alpha)(\varphi\,.\,\psi)\urcorner$	(216)
(2):	$\ulcorner ((\alpha)(\varphi \supset \psi)\,.\,(\mathrm{E}\alpha)\varphi) \supset (\mathrm{E}\alpha)(\varphi\,.\,\psi)\urcorner$	(I, 1, 136)

MT220: $\ulcorner (\mathrm{E}\alpha)(\varphi \vee \psi) \equiv ((\mathrm{E}\alpha)\varphi \vee (\mathrm{E}\alpha)\psi)\urcorner$.

MT221: $\ulcorner (\mathrm{E}\alpha)(\varphi \supset \psi) \equiv ((\alpha)\varphi \supset (\mathrm{E}\alpha)\psi)\urcorner$.

MT222: $\ulcorner (\mathrm{E}\alpha)(\varphi\,.\,\psi) \supset ((\mathrm{E}\alpha)\varphi\,.\,(\mathrm{E}\alpha)\psi)\urcorner$.

MT223: $\ulcorner ((\alpha)\varphi \vee (\alpha)\varphi) \supset (\alpha)(\varphi \vee \psi)\urcorner$.

MT224: $\ulcorner (\alpha)(\varphi \supset \psi) \supset ((\mathrm{E}\alpha)\varphi \supset (\mathrm{E}\alpha)\psi)\urcorner$.

MT225: $\ulcorner ((\mathrm{E}\alpha)\varphi \supset (\mathrm{E}\alpha)\psi) \supset (\mathrm{E}\alpha)(\varphi \supset \psi)\urcorner$.

MT226: $\ulcorner (\alpha)(\varphi\,.\,\psi) \equiv (\varphi\,.\,(\alpha)\psi)\urcorner$, where α is not free in φ.

MT227: $\ulcorner (\alpha)(\varphi\,.\,\psi) \equiv (\varphi\,.\,(\alpha)\psi)\urcorner$, where α is not free in ψ.

MT228: $\ulcorner (\mathrm{E}\alpha)(\varphi\,.\,\psi) \equiv (\varphi\,.\,(\mathrm{E}\alpha)\psi)\urcorner$, where α is not free in φ.

MT229: $\ulcorner (\mathrm{E}\alpha)(\varphi\,.\,\psi) \equiv ((\mathrm{E}\alpha)\varphi\,.\,\psi)\urcorner$, where α is not free in ψ.

MT230: $\ulcorner (\alpha)(\varphi \vee \psi) \equiv (\varphi \vee (\alpha)\psi)\urcorner$, where α is not free in φ.

MT231: $\ulcorner (\alpha)(\varphi \vee \psi) \equiv ((\alpha)\varphi \vee \psi)\urcorner$, where α is not free in ψ.

MT232: $\ulcorner (\mathrm{E}\alpha)(\varphi \vee \psi) \equiv (\varphi \vee (\mathrm{E}\alpha)\psi)\urcorner$, where α is not free in φ.

MT233: $\ulcorner (\mathrm{E}\alpha)(\varphi \vee \psi) \equiv ((\mathrm{E}\alpha)\varphi \vee \psi)\urcorner$, where α is not free in ψ.

MT234: $\ulcorner (\alpha)(\varphi \supset \psi) \equiv (\varphi \supset (\alpha)\psi)\urcorner$, where α is not free in φ.

MT235: $\ulcorner (\alpha)(\varphi \supset \psi) \equiv ((\mathrm{E}\alpha)\varphi \supset \psi)\urcorner$, where α is not free in ψ.

MT236: $\ulcorner (\mathrm{E}\alpha)(\varphi \supset \psi) \equiv (\varphi \supset (\mathrm{E}\alpha)\psi)\urcorner$, where α is not free in φ.

MT237: $\ulcorner (\mathrm{E}\alpha)(\varphi \supset \psi) \equiv ((\alpha)\varphi \supset \psi)\urcorner$, where α is not free in ψ.

Proof:

(1):	$\ulcorner (\alpha){\sim}\psi \supset {\sim}\psi \urcorner$	(202)
(2):	$\ulcorner {\sim}\psi \urcorner$	

(3): $\ulcorner(\alpha)\sim\psi\urcorner$ (U, 2)
(4): $\ulcorner\sim\psi \supset (\alpha)\sim\psi\urcorner$ (C, 2, 3, hypothesis)
(5): $\ulcorner(\alpha)\sim\psi \equiv \sim\psi\urcorner$ (A, 1, 4) D4
(6): $\ulcorner 5 \supset ((\text{E}\alpha)\psi \equiv \sim\sim\psi)\urcorner$ (128) D8
(7): $\ulcorner(\text{E}\alpha)\psi \equiv \sim\sim\psi\urcorner$ (5, 6)
(8): $\ulcorner\psi \equiv \sim\sim\psi\urcorner$ (124)
(9): $\ulcorner(\text{E}\alpha)\psi \equiv \psi\urcorner$ (I, 7, 8)
(10): $\ulcorner(\text{E}\alpha)(\varphi \supset \psi) \equiv ((\alpha)\varphi \supset \psi)\urcorner$ (I, 9, 221)

MT238: $\ulcorner\sim(\alpha)(\beta)\varphi \equiv (\text{E}\alpha)(\text{E}\beta)\sim\varphi\urcorner$.
MT239: $\ulcorner\sim(\text{E}\alpha)(\text{E}\beta)\varphi \equiv (\alpha)(\beta)\sim\varphi\urcorner$.
MT240: $\ulcorner(\alpha)(\beta)\varphi \equiv \sim(\text{E}\alpha)(\text{E}\beta)\sim\varphi\urcorner$.
MT241: $\ulcorner(\text{E}\alpha)(\text{E}\beta)\varphi \equiv \sim(\alpha)(\beta)\sim\varphi\urcorner$.
MT242: $\ulcorner(\alpha)(\beta)\varphi \equiv (\beta)(\alpha)\varphi\urcorner$.

Proof:

(1): $\ulcorner(\alpha)(\beta)\varphi\urcorner$
(2): $\ulcorner 1 \supset (\beta)\varphi\urcorner$ (202)
$\supset \varphi\urcorner$ (202)
(3): φ (1, 2)
(4): $\ulcorner(\alpha)\varphi\urcorner$ (U, 3)
(5): $\ulcorner(\beta)(\alpha)\varphi\urcorner$ (U, 4)
(6): $\ulcorner(\alpha)(\beta)\varphi \supset (\beta)(\alpha)\varphi\urcorner$ (C, 1, 5)

Similarly:

(7): $\ulcorner(\beta)(\alpha)\varphi \supset (\alpha)(\beta)\varphi\urcorner$
(8): $\ulcorner(\alpha)(\beta)\varphi \equiv (\beta)(\alpha)\varphi\urcorner$ (A, 6, 7) D4

MT243: $\ulcorner(\text{E}\alpha)(\text{E}\beta)\varphi \equiv (\text{E}\beta)(\text{E}\alpha)\varphi\urcorner$.
MT244: $\ulcorner(\text{E}\alpha)(\beta)\varphi \supset (\beta)(\text{E}\alpha)\varphi\urcorner$.

30. ALTERNATIVE FORMALIZATIONS

We sketch here a few alternative formalizations of our two calculi.

1. In a first alternative formalization of the sentential calculus, due to B. Rosser, '$\sim$' and '.' serve as primitive connectives,

MB100: $\ulcorner\varphi \supset (\varphi \,.\, \varphi)\urcorner$,
MB101: $\ulcorner(\varphi \,.\, \psi) \supset \varphi\urcorner$.
MB102: $\ulcorner(\varphi \supset \psi) \supset (\sim(\psi \,.\, \chi) \supset \sim(\chi \,.\, \varphi))\urcorner$,

serve as metaaxioms, and Detachment serves as rule of deduction. $\ulcorner(\varphi \vee \psi)\urcorner$ may be defined here as $\ulcorner\sim(\sim\varphi \,.\, \sim\psi)\urcorner$.

2. In a second alternative formalization of the sentential calculus, due essentially to Frege and revised by Łukasiewicz, '$\sim$' and '$\supset$' serve as primitive connectives,

MC100: $\ulcorner\varphi \supset (\psi \supset \varphi)\urcorner$,
MC101: $\ulcorner(\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))\urcorner$,

and

MC102: $\ulcorner(\sim\varphi \supset \sim\psi) \supset (\psi \supset \varphi)\urcorner$

serve as metaaxioms, and Detachment serves as rule of Deduction. $\ulcorner(\varphi \vee \psi)\urcorner$ may be defined here as $\ulcorner((\varphi \supset \psi) \supset \psi)\urcorner$.

3. In a third alternative formalization of the sentential calculus, due essentially to J. Nicod and revised by M. Wajsberg, '|' serves as primitive connective,

MD100: $\ulcorner(\varphi \mid (\psi \mid \chi)) \mid ((\omega \mid (\omega \mid \omega)) \mid ((\omega \mid \psi) \mid ((\varphi \mid \omega) \mid (\varphi \mid \omega))))\urcorner$

serves as metaaxiom, and the following variant of Detachment:

R1′: From φ and $\ulcorner\varphi \mid (\psi \mid \chi)\urcorner$ one may deduce χ,

serves as rule of deduction. $\ulcorner\sim\varphi\urcorner$ and $\ulcorner(\varphi \vee \psi)\urcorner$ may respectively be defined here as $\ulcorner(\varphi \mid \varphi)\urcorner$ and $\ulcorner(\sim\varphi \mid \sim\psi)\urcorner$.

4. In a first alternative formalization of the quantificational calculus, due to the German mathematician D. Hilbert, MA100–MA103 and MA200 serve as metaaxioms, Detachment and the following variant of Universalization:

R2′: From $\ulcorner\varphi \supset \psi\urcorner$ one may deduce $\ulcorner\varphi \supset (\alpha)\psi\urcorner$, where α is not free in φ,

serve as rules of deduction.[20] Our metaaxiom MA201 may be proved in Hilbert's version of the quantificational calculus as follows:

MA201: $\ulcorner(\alpha)(\varphi \supset \psi) \supset (\varphi \supset (\alpha)\psi)\urcorner$, where α is not free in φ.

Proof:

(1):	$\ulcorner(\alpha)(\varphi \supset \psi) \supset (\varphi \supset \psi)\urcorner$	(202)
(2):	$\ulcorner((\alpha)(\varphi \supset \psi) \,.\, \varphi \supset \psi\urcorner$	(I, 1, 150)
(3):	$\ulcorner((\alpha)(\varphi \supset \psi) \,.\, \varphi) \supset (\alpha)\psi\urcorner$	(R2′, 2, and hypothesis on φ)
(4):	$\ulcorner(\alpha)(\varphi \supset \psi) \supset (\varphi \supset (\alpha)\psi)\urcorner$	(I, 3, 150)

Our rule of Universalization, R2, may be derived in turn as follows:

R2: If φ is a theorem, so is $\ulcorner(\alpha)\varphi\urcorner$.

Proof: Let ψ contain no free occurrence of α.

(1):	$\ulcorner\varphi \supset ((\psi \vee \sim\psi) \supset \varphi)\urcorner$	(106)
(2):	$\ulcorner(\psi \vee \sim\psi) \supset \varphi\urcorner$	(1 and hypothesis on φ)
(3):	$\ulcorner(\psi \vee \sim\psi) \supset (\alpha)\varphi\urcorner$	(R2′, 2, and hypothesis on ψ)
(4):	$\ulcorner(\alpha)\varphi\urcorner$	(3, 108)

[20]Hilbert, treating 'E' as a primitive sign, also needs a metaaxiom and a rule of deduction governing 'E'; they are superfluous here.

If we combine 3 with 4, we obtain a version of our two calculi which consists of:

(*a*) two metaaxioms, one sentential metaaxiom (MD100) and one quantificational metaaxiom (MA200);

(*b*) two rules of deduction, one sentential rule of deduction (R1′) and one quantificational rule of deduction (R2′).

5. The next two formalizations of the quantificational calculus, due respectively to W. V. Quine and Hao Wang, use the notion of a closure. We defined in chapter two *a closure of a well-formed formula* φ as any result of prefixing to φ universal quantifiers binding all the free argument variables of φ. It is easily seen that a formula φ which contains 1 free argument variable has 1 closure, that a formula φ which contains 2 free argument variables has 2 closures, and, more generally, that a formula φ which contains n free argument variables has $n!$ closures.[21] Among these $n!$ closures let us define as *the Q-closure of* φ the result of prefixing to φ the string of n ($n \geq 0$) universal quantifiers $\ulcorner(\alpha_1)(\alpha_2) \ldots (\alpha_n)\urcorner$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in alphabetic order the first, the second, . . . , and the n-th free argument variable of φ, and define as *the W-closure of* φ the result of prefixing to φ the string of n ($n \geq 0$) universal quantifiers $\ulcorner(\alpha_n) \ldots (\alpha_2)(\alpha_1)\urcorner$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in alphabetic order the first, the second, . . . , and the n-th free argument variable of φ.

In Quine's formalization of the quantificational calculus the Q-closures of MA100–MA103, of MA200, and of

ME201: $\ulcorner\varphi \supset (\alpha)\varphi\urcorner$, where α is not free in φ,
ME202: $\ulcorner(\alpha)(\varphi \supset \psi) \supset ((\alpha)\varphi \supset (\alpha)\psi\urcorner$,
ME203: $\ulcorner(\alpha)(\beta)\varphi \supset (\beta)(\alpha)\varphi\urcorner$,

may serve as metaaxioms, and Detachment serve as rule of deduction. The theorems provable here are the Q-closures of our own theorems.[22]

In Wang's formalization the W-closures of MA100–MA103, MA200, MA201, and ME201 may serve as metaaxioms, and the following variant of Detachment:

R1″: From the W-closure of φ and the W-closure of $\ulcorner\varphi \supset \psi\urcorner$, one may deduce the W-closure of ψ,

serve as rule of deduction. The theorems provable here are the W-closures of our own theorems. The W-closure of ME201, provable with us through R2, is needed by Wang to prove R2.

[21] $n!$, as the reader may know, is $1 \times 2 \times 3 \times \ldots \times n$.

[22] Cf. Quine's *Mathematical Logic*, first edition (1940), chapter two; the 1951 edition uses W-closures. The formalization sketched in the text differs in several respects from Quine's own formalization.

*31. NATURAL DEDUCTIONS

In 1934 the German logician G. Gentzen proposed a version of the sentential and the quantificational calculus in which he:

(*a*) treats the four connectives '$\sim$', '$\vee$', '$.$', and '$\supset$', and the quantifier letter 'E' as primitive signs;

(*b*) uses derivation schemata in lieu of axioms and rules of deduction; and

(*c*) uses derivations from assumption formulae in lieu of proofs.

The resulting calculus is known as *the calculus of natural deductions.*[23]

Gentzen's derivation schemata for the sentential calculus are eight in number. Four of them serve to introduce the connectives '$\sim$', '$\vee$', '$.$', and '$\supset$' into a conclusion; they are called *introduction schemata.* The other four serve to eliminate the four connectives '$\sim$', '$\vee$', '$.$', and '$\supset$' from a premise; they are called *elimination schemata.*

	Introduction schemata	Elimination schemata
Conjunction:	$\dfrac{\varphi \quad \psi}{\ulcorner \varphi\,.\,\psi \urcorner}$	$\dfrac{\ulcorner \varphi\,.\,\psi \urcorner}{\varphi}$ and $\dfrac{\ulcorner \varphi\,.\,\psi \urcorner}{\psi}$
Conditional:	$\begin{array}{c} \underline{\lvert\varphi\rvert} \\ \vdots \\ \psi \\ \hline \ulcorner \varphi \supset \psi \urcorner \end{array}$	$\dfrac{\varphi \quad \ulcorner \varphi \supset \psi \urcorner}{\psi}$
Alternation:	$\dfrac{\varphi}{\ulcorner \varphi \vee \psi \urcorner}$ and $\dfrac{\varphi}{\ulcorner \psi \vee \varphi \urcorner}$	$\begin{array}{ccc} & \underline{\lvert\varphi\rvert} & \underline{\lvert\psi\rvert} \\ & \vdots & \vdots \\ \ulcorner \varphi \vee \psi \urcorner & \chi & \chi \\ \hline & \chi & \end{array}$
Negation:	$\begin{array}{cc} \underline{\lvert\varphi\rvert} & \underline{\lvert\varphi\rvert} \\ \vdots & \vdots \\ \psi & \ulcorner \sim\psi \urcorner \\ \hline \multicolumn{2}{c}{\ulcorner \sim\varphi \urcorner} \end{array}$	$\dfrac{\ulcorner \sim\sim\varphi \urcorner}{\varphi}$

Explanatory remarks:

(*a*) The introduction schemata for '$.$', '$\supset$', '$\vee$', and '$\sim$' will be respectively referred to as CnI, ClI, AI, and NI; the elimination schemata for

[23]Gentzen offered two versions of the calculus of natural deductions; the one we study here is his so-called *version N*, as revised by P. Bernays.

'$.$', '$\supset$', '$\vee$', and '$\sim$' will be respectively referred to as CnE, ClE, AE, and NE.[24]

(*b*) In each schema the formula written under the horizontal bar is said *to be derivable as a conclusion from the formula* (*e*) *written over the bar as assumption formulae.*

(*c*) '$\underline{|\varphi|}$' stands for a derivation with φ as one of its premises and ψ as

$$\begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \psi \end{array}$$

its conclusion.

(*d*) The formula φ in ClI and NI and the two formulae φ and ψ in AE are said *to be discharged as assumption formulae.*

The following three conventions are adopted:

1. *Reflexivity convention* (R): A well-formed formula φ may be derived as a conclusion from φ as an assumption formula.

2. *Expansion convention* (E): A well-formed formula φ may be inserted before the assumption formula(e) of any derivation schema.

3. *Permutation convention* (P): The assumption formulae of a derivation schema may be permuted at will.

Derivations are set up by Gentzen in so-called *tree-form* as in the following sample:

D1:

$$\dfrac{\dfrac{\overset{3\surd}{p \vee q} \quad \dfrac{\overset{1\surd}{p}}{q \vee p}\mathrm{AI} \quad \dfrac{\overset{2\surd}{q}}{q \vee p}\mathrm{AI} \quad \vee\mathrm{I}}{q \vee p}\mathrm{AE}_{-1,2} \quad \vee\mathrm{E}}{(p \vee q) \supset q(\vee p)}\mathrm{ClI}_{-3} \quad \supset\mathrm{I}$$

Explanation:

(*a*) This derivation starts from three assumption formulae: 'p', 'q', and '$p \vee q$', respectively numbered 1, 2, and 3.

(*b*) From the assumption formula 'p' is derived the conclusion '$q \vee p$', a step justified by the appended reference to AI.

(*c*) From the assumption formula 'q' is derived the conclusion '$q \vee p$', a step justified by the appended reference to AI.

(*d*) From (*b*), (*c*), and the assumption formula '$p \vee q$' is derived the

[24] CnI corresponds to our rule of Adjunction, ClI to our rule of Conditionalization, and ClE to our rule of Detachment.

conclusion '$q \vee p$', a step justified by the appended reference to AE; the two assumption formulae 'p' and 'q' being discharged by AE, we add the subscript '$-1,2$' to the two letters 'AE' and write a check mark after the reference numerals of 'p' and 'q'.

(*e*) From the assumption formula '$p \vee q$' and the conclusion '$q \vee p$' is derived the conclusion '$(p \vee q) \supset (q \vee p)$', a step justified by the appended reference to ClI; the assumption formula '$p \vee q$' being discharged by ClI, we add the subscript '-3' to the three letters 'ClI' and write a check mark after the reference numeral of '$p \vee q$'.

We shall say that a derivation is *completed* if all its assumption formulae have been discharged by ClI, NI, or AE, and we shall say that a well-formed formula φ is a *theorem* if it is the last line of a completed derivation. The above formula '$(p \vee q) \supset (q \vee p)$', being the last line of a completed derivation, may, for instance, be said to be a theorem.

We append a few extra derivations as samples:

D2:

$$\dfrac{\dfrac{\dfrac{\overset{1\checkmark}{q}\quad \overset{1\checkmark}{p}}{p \cdot q}\,\mathrm{CnI}}{q \supset (p \cdot q)}\,\mathrm{ClI}_{-2}}{p \supset (q \supset (p \cdot q))}\,\mathrm{ClI}_{-1}$$

D3:

$$\dfrac{\dfrac{\overset{1\checkmark}{p}}{p \vee q}\,\mathrm{AI}}{p \supset (p \vee q)}\,\mathrm{ClI}_{-1}$$

D4:

$$\dfrac{\dfrac{\overset{2\checkmark}{p \vee p}\quad \dfrac{\overset{1\checkmark}{p}}{p}\,(\mathrm{R})\quad \dfrac{\overset{1\checkmark}{p}}{p}\,(\mathrm{R})}{p}\,\mathrm{AE}_{-1}}{(p \vee p) \supset p}\,\mathrm{ClI}_{-2}$$

Note in D4 the use of convention R, as indicated by '(R)'.

D5:

$$\dfrac{\dfrac{\overset{1\checkmark}{p \cdot \sim p}}{p}\,\mathrm{CnE}\quad \dfrac{\overset{1\checkmark}{p \cdot \sim p}}{\sim p}\,\mathrm{CnE}}{\sim(p \cdot \sim p)}\,\mathrm{NI}_{-1}$$

D6:

$$\dfrac{\dfrac{\overset{1\checkmark}{\sim\sim p}}{p}\,\mathrm{NE}}{\sim\sim p \supset p}\,\mathrm{ClI}_{-1}$$

D7:

$$\cfrac{\cfrac{\cfrac{\overset{1\surd}{p \supset q} \qquad \overset{4\surd}{r \lor p} \qquad \cfrac{\overset{1\surd}{p \supset q} \quad \overset{3\surd}{r}}{r \lor q}\,\text{AI(E)} \qquad \cfrac{\cfrac{\overset{1\surd}{p \supset q} \quad \overset{2\surd}{p}}{q}\,\text{ClE}}{r \lor q}\,\text{AI}}{r \lor q}\,\text{AE}_{-3,2}\text{(E)}}{(r \lor p) \supset (r \lor q)}\,\text{ClI}_{-4}}{(p \supset q) \supset ((r \lor p) \quad (r \lor q))}\,\text{ClI}_{-1}$$

Note in D7 the use of convention E, as indicated by '(E)'.

Gentzen treats $\ulcorner \varphi \equiv \psi \urcorner$ as a paraphrase of $\ulcorner (\varphi \supset \psi) \,.\, (\psi \supset \varphi) \urcorner$; one may also treat '$\equiv$' as a primitive sign and add to the above table the following two derivation schemata:

BI: $$\begin{array}{cc} |\varphi| & |\psi| \\ \vdots & \vdots \\ \psi & \varphi \\ \hline \multicolumn{2}{c}{\ulcorner \varphi \equiv \psi \urcorner} \end{array}$$

BE: $$\frac{\varphi \quad \ulcorner \varphi \equiv \psi \urcorner}{\psi} \text{ and } \frac{\varphi \quad \ulcorner \psi \equiv \varphi \urcorner}{\psi}$$

where the two formulae φ and ψ in BI are said *to be discharged as assumption formulae.* It is possible to derive through use of the resulting ten derivation schemata biconditionals corresponding to D1, D2, and D4, namely:

$$\ulcorner (\varphi \,.\, \psi) \equiv \sim(\sim\varphi \lor \sim\psi) \urcorner,$$

$$\ulcorner (\varphi \supset \psi) \equiv (\sim\varphi \lor \psi) \urcorner,$$

and

$$\ulcorner (\varphi \equiv \psi) \equiv ((\varphi \supset \psi) \,.\, (\psi \supset \varphi)) \urcorner.$$

This task is left to the reader.

Gentzen has offered four derivation schemata governing $\ulcorner (\alpha) \urcorner$ and $\ulcorner (\mathrm{E}\alpha) \urcorner$, namely:

UI: $$\frac{\varphi}{\ulcorner (\alpha)\varphi \urcorner}$$

UE: $$\frac{\ulcorner (\alpha)\varphi \urcorner}{\varphi'}$$

EI: $$\frac{\varphi'}{\ulcorner (\mathrm{E}\alpha)\varphi \urcorner}$$

EE: $$\begin{array}{cc} & |\varphi'| \\ & \vdots \\ \ulcorner (\mathrm{E}\alpha)\varphi \urcorner & \psi \\ \hline \multicolumn{2}{c}{\psi} \end{array}$$

Notes:

(*a*) The formula φ' in EE is said *to be discharged as an assumption formula.*

(*b*) The formula φ' in EI and UE is to be like φ except for containing free occurrences of an argument variable α' wherever φ contains free occurrences of α.

(*c*) The formula φ' in EE is to be like φ except for containing occurrences of an argument variable α' foreign to φ and ψ wherever φ contains free occurrences of α.

(*d*) The argument variable α in UI and EE is not to occur as a free variable in any assumption formula to be discharged by ClI, NI, AE, BI, or EE.

We include three sample derivations:

D8:

$$\dfrac{\dfrac{\overset{1\surd}{(x)F(x)}}{F(y)}\,\text{UE}}{(x)F(x) \supset F(y)}\,\text{ClI}_{-1}$$

D9:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{2\surd}{p} \qquad \dfrac{\overset{1\surd}{(x)(p \supset F(x))}}{p \supset F(x)}\,\text{UE}}{F(x)}\,\text{ClE}}{(x)F(x)}\,\text{UI}}{p \supset (x)F(x)}\,\text{ClI}_{-2}}{(x)(p \supset F(x)) \supset (p \supset (x)F(x))}\,\text{ClI}_{-1}$$

D10:

$$\dfrac{\dfrac{\overset{3\surd}{(\mathrm{E}x)F(x)} \qquad \dfrac{\dfrac{\overset{1\surd}{F(y)}\quad \overset{2\surd}{(x){\sim}F(x)}}{F(y)}\,(\text{R,E,P}) \qquad \dfrac{\overset{1\surd}{F(y)}\quad \overset{2\surd}{(x){\sim}F(x)}}{{\sim}F(y)}\,\text{UE(E)}}{{\sim}(x){\sim}F(x)}\,\text{NI}_{-2}}{{\sim}(x){\sim}F(x)}\,\text{EE}_{-1}}{(\mathrm{E}x)F(x) \supset {\sim}(x){\sim}F(x)}\,\text{ClI}_{-3}$$

Note in D10 the use of conventions R, E, and P. The derivation of the converse conditional: '${\sim}(x){\sim}F(x) \supset (\mathrm{E}x)F(x)$', though long, presents no difficulty.

It could be shown that a well-formed formula φ is derivable as a theorem in Gentzen's version of the sentential or the quantificational calculus if and only if it is provable as a theorem in our own version of that cal-

culus.[25] It could also be shown that if NE is replaced in Gentzen's list of derivation schemata by

$$\text{NE}': \frac{\varphi \quad \ulcorner{\sim}\varphi\urcorner}{\psi},$$

then a well-formed formula φ is derivable as a theorem in the resulting calculus if and only if it is provable as a theorem of Heyting's intuitionist logic.

[25]Gentzen's schemata CnE, CII, AE, NI, and NE could have supplanted in chapter one our rule of Sentential Insertion; similarly, Gentzen's schemata UE, EI, and EE could have supplanted in chapter two our rules of Quantificational Insertion and Relettering.

CHAPTER FOUR

The Logic of Identity, Classes, and Relations

32. IDENTITY

We embark here upon a new task: constructing within the framework of the quantificational calculus a *calculus of identity, classes, and relations.* We shall restrict ourselves in the present section and the next to identity, take up classes in section 34, and finally turn to relations in section 37.

1. *Primitive signs of the identity calculus.* We add to the primitive signs of the quantificational calculus, as given in section 29, paragraph 1, an extra sign:

(*g*) the predicate '$=$',

to be read 'is identical with' or 'is the same as'.

2. *Definition of a formula of the identity calculus*: A formula of the identity calculus is a finite sequence of primitive signs of the identity calculus.

3. *Definition of a well-formed formula of the identity calculus.* We add to the definition of a well-formed formula of the quantificational calculus, as given in section 29, paragraph 3, an extra clause:

(*f*) $\ulcorner(\alpha = \beta)\urcorner$ is a well-formed formula, where α and β are two argument variables.

Note: We shall drop the two parentheses enclosing $\ulcorner\alpha = \beta\urcorner$ in $\ulcorner(\alpha = \beta)\urcorner$ whenever convenient.[1]

4. *Defined signs of the identity calculus.* We add to the eight definitions given in section 29, paragraph 4, a first definition:

D9: $\ulcorner(\alpha \neq \beta)\urcorner \rightarrow \ulcorner\sim(\alpha = \beta)\urcorner$,

where '$\neq$' is interpreted to read 'is distinct from'. We shall introduce 27 more definitions in the course of the chapter. Here as in chapter three we take the liberty of treating as a well-formed formula any paraphrase of a

[1] We shall follow the same policy in a number of cases below without explicit warning.

well-formed formula; '$x \neq y$' and '$x = y \supset y \neq x$', being paraphrases of the two well-formed formulae '$\sim(x = y)$' and '$\sim(x = y) \vee \sim(y = x)$', respectively, will thus count as well-formed formulae.

5. The definitions of a *scope*, of a *bound argument variable*, and of a *free argument variable*, given in section 29, paragraphs 5 and 6, are carried over unchanged.

6. *Definitions of a schema, of a quasi-statement, and of a statement.* All the well-formed formulae of the quantificational calculus were schemata in the sense of chapter one and chapter two. The well-formed formulae of the identity calculus may also be quasi-statements like '$x = x$', '$(y)(x = y \supset y = x)$', and so on, or statements like '$(x)(x = x)$', '$(x)(y)(x = y \supset y = x)$', and so on. We accordingly adopt the following three definitions:

(*a*) A well-formed formula is a *schema* if it contains at least one sentential or one predicate dummy;

(*b*) A well-formed formula is a *quasi-statement* if it contains no sentential or predicate dummy, but contains at least one free argument variable;

(*c*) A well-formed formula is a *statement* if it contains no sentential or predicate dummy and no free argument variable.

In accord with our past policy we shall use the word 'valid' in connection with schemata and quasi-statements, the word 'true' in connection with statements. Identity statements, when true of all universes of discourse, will be logically true; some, like '$(x)(x = x) \supset (x)(x = x)$', will be sententially true; others, like '$(x)(x = x) \supset (\mathrm{E}x)(x = x)$', will be quantificationally true; others, finally, like '$(x)(x = x)$', will be what we may call *identically true.*

7. *Metaaxioms and rules of deduction.* We add to the six metaaxioms MA100–MA103 and MA200–MA201, given in section 29, paragraph 7, two metaaxioms governing '$=$'; the first, called *the law of reflexivity for* '$=$', states that everything is self-identical; the second, called *the law of substitutivity for* '$=$', states that if two entities are identical, then whatever is true of the first is true of the second and *vice-versa.*

MA300: $\ulcorner(\alpha)(\alpha = \alpha)\urcorner$.
MA301: $\ulcorner\alpha = \beta \supset (\varphi \equiv \varphi')\urcorner$, where φ' is like φ except for containing free occurrences of β in place of some free occurrences of α.

Detachment (R1) and Universalization (R2) are carried over as rules of deduction.

Notes:

(*a*) MA300 could be replaced by the metaaxiom $\ulcorner\alpha = \alpha\urcorner$ (from which MA300 follows by R2), or by either one of the two axioms '$(x)(x = x)$'

(from which MA300 follows by R7) and '$x = x$' (from which MA300 follows by R7 and R2).

(*b*) MA301 could be replaced by the following metaaxiom:

MA301′: $\ulcorner \alpha = \beta \supset (\varphi \supset \varphi') \urcorner$, where φ' is like φ except for containing one free occurrence of β in place of one free occurrence of α.

The more advanced reader may undertake the proof of MA301 from MA301′.

8. The definitions of a *proof*, of a *theorem*, of a *derivation from assumption formulae*, and of a *formula derivable from assumption formulae*, given in section 29, paragraphs 8 and 10, are carried over unchanged.

9. *Sample theorems and metatheorems.* It can be shown that MA300 and MA301 yield as theorems all valid, as well as all logically true, formulae of the identity calculus. A few well-known formulae are proved here as samples.

T302: $(x)(\mathrm{E}y)(x = y)$.

Proof:

(1):	$(x)(x = x) \supset x = x$	(202)
	$\supset (\mathrm{E}y)(x = y)$	(217)
(2):	$(\mathrm{E}y)(x = y)$	(300, 1)
(3):	$(x)(\mathrm{E}y)(x = y)$	(U, 2)

According to T302 everything is identical with something or other.

T303: $(x)(y)(x = y \equiv y = x)$.

Proof:

(1):	$x = y \supset (x = x \equiv y = x)$	(301)
	$\supset (x = x \supset y = x)$	(156)
(2):	$x = x \supset (x = y \supset y = x)$	(I, 119, 1)
(3):	$(x)(x = x) \supset x = x$	(202)
	$\supset (x = y \supset y = x)$	(2)
(4):	$x = y \supset y = x$	(300, 3)
Similarly:		
(5):	$y = x \supset x = y$	
(6):	$x = y \equiv y = x$	(A, 4, 5) D4
(7):	$(x)(y)(x = y \equiv y = x)$	(U, (U, 6))

T303 is called *the law of symmetry* (or *commutativity*) *for* '='.

T304: $(x)(y)(z)((x = y \,.\, y = z) \supset x = z)$.

Proof:

(1):	$x = y \supset (x = z \equiv y = z)$	(301)
(2):	$x = y \supset (y = z \equiv x = z)$	(I, 138, 1)
	$\supset (y = z \supset x = z)$	(156)

(3): $(x = y \,.\, y = z) \supset x = z$ (I, 150, 2)
(4): $(x)(y)(z)((x = y \,.\, y = z) \supset x = z)$ (U, (U, (U, 3)))

T304 is called *the law of transivity for '='*. A variant of it is appended according to which two entities are identical if they are both identical with a third one.

T305: $(x)(y)(z)((x = y \,.\, z = y) \supset x = z)$.

The proof of T305 is left to the reader.[2]

MT306: $\ulcorner \varphi' \equiv (\beta)(\beta = \alpha \supset \varphi)\urcorner$, where α is not β and φ' is like φ except for containing free occurrences of α wherever φ contains free occurrences of β.

Proof:

(1):	$\ulcorner(\beta)(\beta = \alpha \supset \varphi) \supset (\alpha = \alpha \supset \varphi')\urcorner$	(200 and hypothesis on φ')
(2):	$\ulcorner\alpha = \alpha \supset ((\beta)(\beta = \alpha \supset \varphi) \supset \varphi')\urcorner$	(I, 119, 1)
(3):	$\ulcorner 300 \supset \alpha = \alpha\urcorner$	(202)
	$\supset ((\beta)(\beta = \alpha \supset \varphi) \supset \varphi')\urcorner$	(2)
(4):	$\ulcorner(\beta)(\beta = \alpha \supset \varphi) \supset \varphi\urcorner$	(300, 3)
(5):	$\ulcorner\beta = \alpha \supset (\varphi \equiv \varphi')\urcorner$	(301 and hypothesis on φ')
(6):	$\ulcorner\beta = \alpha \supset (\varphi' \equiv \varphi)\urcorner$	(I, 138, 5)
	$\supset (\varphi' \supset \varphi)\urcorner$	(156)
(7):	$\ulcorner\varphi' \supset (\beta = \alpha \supset \varphi)\urcorner$	(I, 119, 6)
(8):	$\ulcorner(\beta)7 \supset (\varphi' \supset (\beta)(\beta = \alpha \supset \varphi))\urcorner$	(201 and hypotheses)
(9):	$\ulcorner\varphi' \supset (\beta)(\beta = \alpha \supset \varphi)\urcorner$	((U, 7), 8)
(10):	$\ulcorner\varphi' \equiv (\beta)(\beta = \alpha \supset \varphi)\urcorner$	(A, 4, 9) D4

In virtue of MT306 such biconditionals as

John loves ice cream $\equiv (x)(x =$ John $\supset x$ loves ice cream),
Cicero was a Roman $\equiv (w)(w =$ Cicero $\supset w$ was a Roman),

and so on, are logically true.

MT307: $\ulcorner\varphi' \equiv (\mathrm{E}\beta)(\beta = \alpha \,.\, \varphi)\urcorner$, where α and φ' are as in MT306.
MT308: $\ulcorner(\varphi \,.\, \sim\varphi') \supset \alpha \neq \beta\urcorner$, where φ' is like φ except for containing some free occurrences of β in place of some free occurrences of α.

33. NUMERICAL QUANTIFIERS

We distinguished in chapter two between three sets of quantifiers: 'There exist at least n x such that', 'There exist at most n x such that', and 'There exist exactly n x such that', where $n = 1, 2, 3, \ldots$. Having drawn the distinction, we restricted ourselves, however, to one quantifier,

[2]The same remark applies to all further theorems or metatheorems recorded without proofs.

'There exists at least one x such that'. A general account of the so-called *numerical quantifiers* may now be given.

Let us first abbreviate 'There exist at least n x such that $F(x)$' as '$(\mathrm{EL}_{n}x)F(x)$', 'There exist at most n x such that $F(x)$' as '$(\mathrm{EM}_{n}x)F(x)$', and 'There exist exactly n x such that $F(x)$' as '$(\mathrm{EE}_{n}x)F(x)$'.

1. '$(\mathrm{EM}_{n}x)F(x)$' is readily defined as '$\sim(\mathrm{EL}_{n+1}x)F(x)$'; to say, for example, that a student may register in at most one course is to say that he may not register in two courses or more; to say that he may register in at most two courses is to say that he may not register in three courses or more, and so on.

2. '$(\mathrm{EE}_{n}x)F(x)$' is readily defined in turn as '$(\mathrm{EL}_{n}x)F(x) \,.\, (\mathrm{EM}_{n}x)F(x)$'; to say, for example, that a student may register in exactly one course is to say that he may register in at least one and may register in at most one course.

3. The only problem left is thus to define '$(\mathrm{EL}_{n}x)F(x)$'.

(*a*) '$(\mathrm{EL}_{1}x)F(x)$' is already available as '$(\mathrm{E}x)F(x)$';

(*b*) To say that a predicate is true of at least $n + 1$ entities is to say that the predicate in question is true of at least one entity and also true of at least n entities distinct from the previous one; '$(\mathrm{EL}_{n+1}x)F(x)$' may accordingly be defined as '$(\mathrm{E}x)(F(x) \,.\, (\mathrm{EL}_{n}y)(F(y) \,.\, y \neq x))$'.

The two clauses (*a*) and (*b*) constitute what we call a *recursive definition* of '$(\mathrm{EL}_{n}x)F(x)$'.

We restate the present definitions to read:

D10a: $\ulcorner(\mathrm{EL}_{1}\alpha)\varphi\urcorner \rightarrow \ulcorner(\mathrm{E}\alpha)\varphi\urcorner$;

D10b: $\ulcorner(\mathrm{EL}_{n+1}\alpha)\varphi\urcorner \rightarrow \ulcorner(\mathrm{E}\alpha)(\varphi \,.\, (\mathrm{EL}_{n}\beta)(\varphi' \,.\, \beta \neq \alpha))\urcorner$, where β is foreign to φ and φ' is like φ except for containing occurrences of β wherever φ contains free occurrences of α.[3]

D11: $\ulcorner(\mathrm{EM}_{n}\alpha)\varphi\urcorner \rightarrow \ulcorner\sim(\mathrm{EL}_{n+1}\alpha)\varphi\urcorner$.

D12: $\ulcorner(\mathrm{EE}_{n}\alpha)\varphi\urcorner \rightarrow \ulcorner(\mathrm{EL}_{n}\alpha)\varphi \,.\, (\mathrm{EM}_{n}\alpha)\varphi\urcorner$.

[3] However numerous the variables of φ may be, there is still an infinite number of variables which are foreign to φ and qualify as the β of D10b. $\ulcorner(\mathrm{EL}_{n+1}\alpha)\varphi\urcorner$ will thus have an infinite number of *definientia*, all of which are of course equivalent. We could easily cut that number of *definientia* down to 1 by requiring β to be the alphabetically first variable foreign to φ.

According to D10a–D10b '$(\mathrm{EL}_{2}x)F(x)$' is a paraphrase of '$(\mathrm{E}x)(F(x) \,.\, (\mathrm{EL}_{1}y)(F(y) \,.\, y \neq x))$', that is, '$(\mathrm{E}x)(F(x) \,.\, (\mathrm{E}y)(F(y) \,.\, y \neq x))$'; '$(\mathrm{EL}_{3}x)F(x)$' is a paraphrase of '$(\mathrm{E}x)(F(x) \,.\, (\mathrm{EL}_{2}y)(F(y) \,.\, y \neq x))$', that is, '$(\mathrm{E}x)(F(x) \,.\, (\mathrm{E}y)((F(y) \,.\, y \neq x) \,.\, (\mathrm{EL}_{1}z)((F(z) \,.\, z \neq x) \,.\, z \neq y)))$', that is, '$(\mathrm{E}x)(F(x) \,.\, (\mathrm{E}y)((F(y) \,.\, y \neq x) \,.\, (\mathrm{E}z)((F(z) \,.\, z \neq x) \,.\, z \neq y)))$'; and so on.

We next list sample metatheorems about 'EL_{n}', 'EM_{n}', and 'EE_{n}'. According to MT320 'at least $n + 1$' implies 'at least n'.

MT320: $\ulcorner(\mathrm{EL}_{n+1}\alpha)\varphi \supset (\mathrm{EL}_{n}\alpha)\varphi\urcorner$.

Proof: Let β be foreign to φ and let φ' be like φ except for containing occurrences of β wherever φ contains free occurrences of α.

Step 1: We first show that MT320 holds when $n = 1$.

$\ulcorner(\mathrm{EL}_{2}\alpha)\varphi \supset ((\mathrm{E}\alpha)\varphi \,.\, (\mathrm{E}\alpha)(\mathrm{E}\beta)(\varphi' \,.\, \beta \neq \alpha))\urcorner$ (222) D10b, D10a

$\supset (\mathrm{EL}_{1}\alpha)\varphi\urcorner$ (126) D10a

Step 2: We next show that if MT320 holds for a given natural number m, it also holds for $m + 1$.

(1): $\ulcorner(\mathrm{EL}_{m}\beta)(\varphi' \,.\, \beta \neq \alpha) \supset (\mathrm{EL}_{m-1}\beta)(\varphi' \,.\, \beta \neq \alpha)\urcorner$ (by hypothesis on m)

(2): $\ulcorner(\varphi \,.\, (\mathrm{EL}_{m}\beta)(\varphi' \,.\, \beta \neq \alpha)) \supset (\varphi \,.\, (\mathrm{EL}_{m-1}\beta)(\varphi' \,.\, \beta \neq \alpha))\urcorner$ (152, 1)

(3): $\ulcorner(\mathrm{EL}_{m+1}\alpha)\varphi \supset (\mathrm{EL}_{m}\alpha)\varphi\urcorner$ ((U, 2), 224) D10b

Metatheorem MT320 follows from step 1 and step 2 by mathematical induction.

According to the next metatheorem 'at most n' implies 'at most $n + 1$':

MT321: $\ulcorner(\mathrm{EM}_{n}\alpha)\varphi \supset (\mathrm{EM}_{n+1}\alpha)\varphi\urcorner$

Proof: From MT320 by MT153 and D11.

The following three metatheorems provide alternative translations of $\ulcorner(\mathrm{EL}_{n}\alpha)\varphi\urcorner$, $\ulcorner(\mathrm{EM}_{n}\alpha)\varphi\urcorner$, and $\ulcorner(\mathrm{EE}_{n}\alpha)\varphi\urcorner$. To facilitate reading we omit a few easily restored parentheses.

MT322: $\ulcorner(\mathrm{EL}_{n}\alpha)\varphi \equiv (\mathrm{E}\alpha)(\mathrm{E}\beta_1) \ldots (\mathrm{E}\beta_{n-1})((\varphi \,.\, \varphi_1 \,.\, \ldots \,.\, \varphi_{n-1}) \,.\, (\alpha \neq \beta_1 \,.\, \alpha \neq \beta_2 \,.\, \ldots \,.\, \alpha \neq \beta_{n-1} \,.\, \beta_1 \neq \beta_2 \,.\, \ldots \,.\, \beta_1 \neq \beta_{n-1} \,.\, \ldots \,.\, \beta_{n-2} \neq$

$\beta_{n-1}))^\urcorner$, where $n \geq 1, \beta_1, \beta_2, \ldots, \beta_{n-1}$ are distinct argument variables not free in φ, and φ_i is like φ except for containing free occurrences of β_i wherever φ contains free occurrences of α.

For instance:

$$(\underset{2}{\mathrm{EL}}x)F(x) \equiv (\mathrm{E}x)(\mathrm{E}y)(F(x) \,.\, F(y) \,.\, x \neq y),$$

$$(\underset{3}{\mathrm{EL}}x)F(x) \equiv (\mathrm{E}x)(\mathrm{E}y)(\mathrm{E}z)(F(x) \,.\, F(y) \,.\, F(z) \,.\, x \neq y \,.\, x \neq z \,.\, y \neq z),$$

and so on.

MT323: $^\ulcorner(\underset{n}{\mathrm{EM}}\alpha)\varphi \equiv (\alpha)(\beta_1) \ldots (\beta_n)((\varphi \,.\, \varphi_1 \,.\, \ldots \,.\, \varphi_n) \supset (\alpha = \beta_1 \vee \alpha = \beta_2 \vee \ldots \vee \alpha = \beta_n \vee \beta_1 = \beta_2 \vee \ldots \vee \beta_1 = \beta_n \vee \ldots \vee \beta_{n-1} = \beta_n))^\urcorner$, where $n, \beta_1, \beta_2, \ldots, \beta_n$, and φ_i are as in MT322.

For instance:

$$(\underset{1}{\mathrm{EM}}x)F(x) \equiv (x)(y)((F(x) \,.\, F(y)) \supset x = y),$$

$$(\underset{2}{\mathrm{EM}}x)F(x) \equiv (x)(y)(z)((F(x) \,.\, F(y) \,.\, F(z)) \supset (x = y \vee x = z \vee y = z)),$$

and so on.

MT324: $^\ulcorner(\underset{n}{\mathrm{EE}}\alpha)\varphi \equiv (\mathrm{E}\alpha)(\mathrm{E}\beta_1) \ldots (\mathrm{E}\beta_{n-1})((\alpha \neq \beta_1 \,.\, \alpha \neq \beta_2 \,.\, \ldots \,.\, \alpha \neq \beta_{n-1} \,.\, \beta_1 \neq \beta_2 \,.\, \ldots \,.\, \beta_1 \neq \beta_{n-1} \,.\, \ldots \,.\, \beta_{n-2} \neq \beta_{n-1}) \,.\, (\beta_n)(\varphi_n \equiv (\beta_n = \alpha \vee \beta_n = \beta_1 \vee \ldots \vee \beta_n = \beta_{n-1})))^\urcorner$, where $n, \beta_1, \beta_2, \ldots, \beta_n$, and φ_n are as in MT322.

For instance:

$$(\underset{1}{\mathrm{EE}}x)F(x) \equiv (\mathrm{E}x)(y)(F(y) \equiv x = y),$$

$$(\underset{2}{\mathrm{EE}}x)F(x) \equiv (\mathrm{E}x)(\mathrm{E}y)(x \neq y \,.\, (z)(F(z) \equiv (z = x \vee z = y))),$$

and so on.

According to MT323, to say that a predicate is true of at most n entities is to say that if the predicate is true of $n + 1$ entities, then at least two of these entities are identical with one another. According to MT324, to say that a predicate is true of exactly n entities is to say that the predicate is true of a given entity if and only if that entity is identical with one or the other of n distinct entities.

34. CLASSES AND ONE-PLACE ABSTRACTS

The various entities of which a monadic predicate '—— ——' is true may be regarded as forming a class, the class of all x's such that —— x ——, called *the extension of* '—— ——'. The various entities of

which the predicate 'is a man' is true, for instance, may be regarded as forming a class, the class of all men; similarly, the various entities of which the predicate 'is a natural number' is true may be regarded as forming a class, the class of all natural numbers. Now it is clear that a given entity x will belong to a class if and only if the predicate of which the class is the extension is true of x. For instance, a given entity x will belong to the class of all men if and only if the predicate 'is a man' is true of x; similarly, a given entity x will belong to the class of all natural numbers if and only if the predicate 'is a natural number' is true of x. If we abbreviate 'belongs to' as 'ϵ' and 'the class of all y's such that $F(y)$' as '$\hat{y}F(y)$', we may accordingly treat the formula '$x \epsilon \hat{y}F(y)$' as a paraphrase of the formula '$F(x)$'.[3a]

Generalizing upon this remark, we adopt the following definition:

D13: Let $\varphi_{-\beta}$ be the result of replacing all the bound occurrences of β in φ by occurrences of an argument variable foreign to φ; then $\ulcorner(\beta \epsilon \hat{\alpha}\varphi)\urcorner \rightarrow \varphi'$, where φ' is like $\varphi_{-\beta}$ except for containing occurrences of β wherever $\varphi_{-\beta}$ contains free occurrences of α.[4]

D13 introduces two new signs into the identity calculus:

(*a*) the letter 'ϵ', borrowed from the Greek word 'ἐστί' and read 'belongs to' or 'is a member of';

(*b*) the circumflex accent '^', read 'the class of all . . .'s such that'.

The formulae $\ulcorner\hat{\alpha}\varphi\urcorner$ will be called *one-place* or *class abstracts*; the formulae $\ulcorner(\beta \epsilon \hat{\alpha}\varphi)\urcorner$, *class formulae.*

As a predicate dummy, say 'F', serves as a place-holder for predicates, so the abstract '$\hat{x}F(x)$' will serve as a place-holder for the various class names which result from substituting predicates for 'F' in '$\hat{x}F(x)$'; it will, for example, serve as a place-holder for the class names '$\hat{x}$(x is a man)', '$\hat{x}$(x is a natural number)', and so on, which respectively result from substituting 'is a man', 'is a natural number', and so on, for 'F' in '$\hat{x}F(x)$'.

[3a]In mathematical writings a class is often called a *set*; we shall use the two words interchangeably.

[4]The simpler definition:

D13: $\ulcorner(\beta \epsilon \hat{\alpha}\varphi)\urcorner \rightarrow \varphi'$, where φ' is like φ except for containing free occurrences of β wherever φ contains free occurrences of α,

fails to account for such formulae as '$y \epsilon \hat{x}(y)\mathrm{F}(x,y)$', '$y \epsilon \hat{x}(\mathrm{E}y)\mathrm{F}(x,y)$', and so on, where β is already bound in φ. There is indeed no formula φ' which is like '$(y)\mathrm{F}(x,y)$' or '$(\mathrm{E}y)\mathrm{F}(x,y)$' except for containing free occurrences of 'y' wherever '$(y)\mathrm{F}(x,y)$' or '$(\mathrm{E}y)\mathrm{F}(x,y)$' contains free occurrences of 'x'. According to D13, on the other hand, '$y \epsilon \hat{x}(y)\mathrm{F}(x,y)$' stands for '$(w)\mathrm{F}(y,w)$', or '$(z)\mathrm{F}(y,z)$', and so on, and '$y \epsilon \hat{x}(\mathrm{E}y)\mathrm{F}(x,y)$' stands for '$(\mathrm{E}w)\mathrm{F}(y,w)$', or '$(\mathrm{E}z)\mathrm{F}(y,z)$', and so on. D13 will be forthcoming as a metatheorem (MT331).

With D13 at hand, we may introduce into the identity calculus a number of additional class formulae; these class formulae make up what is sometimes called *the elementary calculus of classes.* In section 37 we shall show that the relation formula 'w bears to z the relation of all x's to all y's such that $F(x,y)$', in symbols: '$w(\hat{x}\hat{y}F(x,y))z$', may serve as a paraphrase of the formula '$F(w,z)$' (definition D23). With D23 at hand, we shall introduce into the identity calculus a number of additional relation formulae; these relation formulae will make up what is sometimes called *the elementary calculus of relations.*

The elementary calculus of classes and its mate, the elementary calculus of relations, are fragments of a wider calculus, called *set theory.*[5] Set theory falls outside the scope of this book; it differs, we may note, from the above calculi in two main respects:

(a) its argument variables may take individuals, classes, classes of classes, and so on, as their values;[6]

(b) its membership predicate 'ϵ' may be flanked on both sides by argument variables.

Definitions D14–D21 introduce eight extra class formulae into the identity calculus.

(a) Two classes $\hat{x}F(x)$ and $\hat{y}G(y)$ will be said *to be identical,* in symbols:

$$\hat{x}F(x) = \hat{y}G(y),$$

when their members are the same, in symbols:

$$(z)(z \in \hat{x}F(x) \equiv z \in \hat{y}G(y));$$

the class of all vegetarians, for instance, is identical with the class of all plant-eaters.

D14: $\ulcorner(\hat{\alpha}\varphi = \hat{\beta}\psi)\urcorner \rightarrow \ulcorner(\gamma)(\gamma \in \hat{\alpha}\varphi \equiv \gamma \in \hat{\beta}\psi)\urcorner$, where γ is new.

(b) Two classes $\hat{x}F(x)$ and $\hat{y}G(y)$ will be said *to be distinct,* in symbols:

$$\hat{x}F(x) \neq \hat{y}G(y),$$

when they are not identical, in symbols:

$$\sim(\hat{x}F(x) = \hat{y}G(y));$$

[5]Set theory may be formalized as an independent calculus or it may be treated as a fragment of the so-called *higher quantificational calculus.* Early versions of it were plagued by contradictions, the famous *logical paradoxes* of Cantor, Burali-Forti, and Russell. It is to set theory, let us note, that mathematics reduces (cf. section 4).

[6]Relations are definable in set theory as classes of a sort and hence need not be assumed explicitly among the values of the argument variables 'w', 'x', 'y', and 'z'.

the class of all vegetarians, for instance, is distinct from the class of all meat-eaters.

D15: $\ulcorner(\hat{\alpha}\varphi \neq \hat{\beta}\psi)\urcorner \rightarrow \ulcorner\sim(\hat{\alpha}\varphi = \hat{\beta}\psi)\urcorner$.[7]

(*c*) A class $\hat{x}F(x)$ will be said *to be included in* or *to be a subclass of* another class $\hat{y}G(y)$, in symbols:

$$\hat{x}F(x) \subset \hat{y}G(y),$$

when all the members of the first class are members of the second, in symbols:

$$(z)(z \in \hat{x}F(x) \supset z \in \hat{y}G(y));$$

the class of all Catholics, for instance, is included in the class of all Christians.

D16: $\ulcorner(\hat{\alpha}\varphi \subset \hat{\beta}\psi)\urcorner \rightarrow \ulcorner(\gamma)(\gamma \in \hat{\alpha}\varphi \supset \gamma \in \hat{\beta}\psi)\urcorner$, where γ is new.[8]

(*d*) By *the sum of two classes* $\hat{x}F(x)$ and $\hat{y}G(y)$, in symbols:

$$\hat{x}F(x) \cup \hat{y}G(y),$$

we shall understand the class of all entities which belong to at least one of the two classes, in symbols:

$$\hat{z}(z \in \hat{x}F(x) \vee z \in \hat{y}G(y));$$

the class of all upperclassmen, for instance, is the sum of the class of all juniors and the class of all seniors.

D17: $\ulcorner(\hat{\alpha}\varphi \cup \hat{\beta}\psi)\urcorner \rightarrow \ulcorner\hat{\gamma}(\gamma \in \hat{\alpha}\varphi \vee \gamma \in \hat{\beta}\psi)\urcorner$, where γ is new.[8a]

(*e*) By *the product of two classes* $\hat{x}F(x)$ and $\hat{y}G(y)$, in symbols:

$$\hat{x}F(x) \cap \hat{y}G(y),$$

we shall understand the class of all entities which belong to both classes, in symbols:

$$\hat{z}(z \in \hat{x}F(x) \,.\, z \in \hat{y}G(y));$$

the class of all female college teachers, for instance, is the product of the class of all females and the class of all college teachers.

[7]The class predicates '$=$' and '$\neq$' are easily told from their non-class analogues by the expressions they govern: class abstracts as opposed to argument variables.

[8]The class predicate '$\subset$' is easily told from the sentential connective '$\subset$' (D3) by the expressions it governs: class abstracts as opposed to well-formed formulae. Note that in mathematical writings a subclass is often called a *subset*.

[8a]In mathematical writings the sum of two classes is often called *the union of two classes*.

D18: $\ulcorner(\hat{\alpha}\varphi \cap \hat{\beta}\psi)\urcorner \rightarrow \ulcorner\hat{\gamma}(\gamma \in \hat{\alpha}\varphi \,.\, \gamma \in \hat{\beta}\psi)\urcorner$, where γ is new.[8b]

(*f*) By *the complement of a class* $\hat{x}F(x)$, in symbols:

$$\overline{\hat{x}F(x)},$$

we shall understand the class of all entities which do not belong to $\hat{x}F(x)$, in symbols:

$$\hat{y}\sim(y \in \hat{x}F(x));$$

the class of all inorganic entities, for instance, is the complement of the class of all organic entities.

D19: $\ulcorner\overline{\hat{\alpha}\varphi}\urcorner \rightarrow \ulcorner\hat{\gamma}\sim(\gamma \in \hat{\alpha}\varphi)\urcorner$, where γ is new.

(*g*) The predicate of which a class is the extension may, like the predicate 'is not self-identical', be true of nothing, or, like the predicate 'is a man', be true of something short of everything, or, like the predicate 'is self-identical', be true of everything. By *the null class* Λ we shall understand the extension of the predicate 'is not self-identical', by *the universal class* V the extension of the predicate 'is self-identical'.

D20: 'Λ' $\rightarrow$ '$\hat{x}(x \neq x)$';
D21: 'V' $\rightarrow$ '$\hat{x}(x = x)$'.

D13–D19 are contextual definitions, D20 and D21 explicit ones.

We now list a series of metatheorems concerning the above class signs; many of them are left without proof.

MT330: $\ulcorner\beta \in \hat{\alpha}\varphi \equiv \varphi'\urcorner$, where β is not bound in φ and φ' is like φ except for containing occurrences of β wherever φ contains free occurrences of α.

Proof: By MT118, D13, and hypotheses.

MT331: $\ulcorner\beta \in \hat{\alpha}\varphi \equiv \varphi'\urcorner$, where φ' is like φ except for containing free occurrences of β wherever φ contains free occurrences of α.

Proof: By MT330.

MT332: $\ulcorner\alpha \in \hat{\alpha}\varphi \equiv \varphi\urcorner$.

Proof: By MT331 where β is α.

MT333: $\ulcorner\hat{\alpha}\varphi = \hat{\beta}(\beta \in \hat{\alpha}\varphi)\urcorner$, where β is not free in φ.

Proof: Let γ be new.

(1): $\ulcorner\gamma \in \hat{\alpha}\varphi \equiv \gamma \in \hat{\beta}(\beta \in \hat{\alpha}\varphi)\urcorner$ (I, 118, 331 and hyp. on β)
(2): $\ulcorner\hat{\alpha}\varphi = \hat{\beta}(\beta \in \hat{\alpha}\varphi)\urcorner$ (U, 1) D14

According to MT333 a class is the class of all its members.

[8b]In mathematical writings the product of two classes is often called *the intersection* or *the meet of two classes*.

Our next metatheorem is *the law of Substitutivity for class identity*:

MT334: $\ulcorner(\alpha_1)(\alpha_2) \ldots (\alpha_n)(\hat{\beta}\varphi = \hat{\gamma}\psi) \supset (\chi \equiv \omega)\urcorner$, where ω is like χ except for containing $\ulcorner\hat{\gamma}\psi\urcorner$ at some places where χ contains $\ulcorner\hat{\beta}\varphi\urcorner$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all the free argument variables of $\ulcorner\hat{\beta}\varphi\urcorner$ and $\ulcorner\hat{\gamma}\psi\urcorner$.

The proof of MT333 is left to the more advanced reader (Hints: 1. Eliminate through D13–D21 all occurrences of $\ulcorner\hat{\beta}\varphi\urcorner$ and $\ulcorner\hat{\gamma}\psi\urcorner$ in $\ulcorner\hat{\beta}\varphi = \hat{\gamma}\psi\urcorner$, χ, and ω; 2. Apply step one of our proof of R6, section 29, to the resulting formulae).

MT335: $\ulcorner\hat{\alpha}\varphi = \hat{\beta}\psi \equiv (\alpha)(\varphi \equiv \psi')\urcorner$, where α does not occur in ψ and ψ' is like ψ except for containing occurrences of α wherever ψ contains free occurrences of β.

Proof: Let γ be new, φ' be like φ except for containing occurrences of γ wherever φ contains free occurrences of α, and ψ'' be like ψ except for containing occurrences of γ wherever ψ contains free occurrences of β.

(1): $\ulcorner\hat{\alpha}\varphi = \hat{\beta}\psi \equiv (\gamma)(\varphi' \equiv \gamma \in \hat{\beta}\psi)\urcorner$	(I, 118, 330) D14
(2): $\ulcorner\hat{\alpha}\varphi = \hat{\beta}\psi \equiv (\gamma)(\varphi' \equiv \psi'')\urcorner$	(I, 118, 330)
(3): $\ulcorner\hat{\alpha}\varphi = \hat{\beta}\psi \equiv (\alpha)(\varphi \equiv \psi')\urcorner$	(R, 2, and hypotheses on α and ψ')

MT336: $\ulcorner\hat{\alpha}\varphi \subset \hat{\beta}\psi \equiv (\alpha)(\varphi \supset \psi')\urcorner$, where α and ψ' are as in MT335.
MT337: $\ulcorner\hat{\alpha}\varphi \cup \hat{\beta}\psi = \hat{\alpha}(\varphi \vee \psi')\urcorner$, where α and ψ' are as in MT335.
MT338: $\ulcorner\hat{\alpha}\varphi \cap \hat{\beta}\psi = \alpha(\varphi \,.\, \psi')\urcorner$, where α and ψ' are as in MT335.
MT339: $\ulcorner\overline{\hat{\alpha}\varphi} = \hat{\alpha}{\sim}\varphi\urcorner$.
MT340: $\ulcorner\beta \in \hat{\alpha}\varphi \cup \hat{\gamma}\psi \equiv (\beta \in \hat{\alpha}\varphi \vee \beta \in \hat{\gamma}\psi)\urcorner$.
MT341: $\ulcorner\beta \in \hat{\alpha}\varphi \cap \hat{\gamma}\psi \equiv (\beta \in \hat{\alpha}\varphi \,.\, \beta \in \hat{\gamma}\psi)\urcorner$.
MT342: $\ulcorner\beta \in \overline{\hat{\alpha}\varphi} \equiv {\sim}(\beta \in \hat{\alpha}\varphi)\urcorner$.
MT343: $\ulcorner\hat{\alpha}\varphi = \Lambda \equiv {\sim}(\mathrm{E}\alpha)\varphi\urcorner$.
MT344: $\ulcorner\hat{\alpha}\varphi = \mathrm{V} \equiv (\alpha)\varphi\urcorner$.

According to MT343 a given class $\hat{x}$(—— x ——) is identical with the null class if and only if '——   ——' is true of nothing; according to MT344 a given class $\hat{x}$(—— x ——) is identical with the universal class if and only if '——   ——' is true of everything.

We close this section with two theorems; according to the first one nothing belongs to Λ, according to the second everything belongs to V.

T345: ${\sim}(\mathrm{E}x)(x \in \Lambda)$.
T346: $(x)(x \in \mathrm{V})$. (I, 332, 300) D21

35. THE BOOLEAN ALGEBRA OF CLASSES

Having at hand the three predicates '=', '≠', and '⊂', the three operators '∪', '∩', and '—', and the two abstracts 'Λ' and 'V', we

may define the notion of a Boolean class formula. A *Boolean class formula* is a well-formed formula of the identity calculus which contains only:

1. sentential connectives;
2. one-place abstracts $\ulcorner \hat{\alpha}\kappa(\alpha) \urcorner$;
3. Boolean signs, that is, '$=$', '$\neq$', '$\subset$', '$\cup$', '$\cap$', '$-$', 'Λ', and/or 'V'.

Boolean class formulae make up a fragment of the elementary calculus of classes known as *the Boolean algebra of classes.*[9]

There exists a mechanical procedure for turning a Boolean class formula φ into a formula φ' of the monadic quantificational calculus called *the monadic associate of* φ. It consists of eight steps to be taken in the following order:

Step 1: Replace each occurrence of 'V' in φ by '$\hat{x}(F(x) \supset F(x))$'; let the resulting formula be φ_1;

Step 2: Replace each occurrence of 'Λ' in φ_1 by '$\hat{x}{\sim}(F(x) \supset F(x))$'; let the resulting formula be φ_2;

Step 3: Replace each occurrence of $\ulcorner \overline{\hat{\alpha}\kappa(\alpha)} \urcorner$ in φ_2 by $\ulcorner \hat{\alpha}{\sim}\kappa(\alpha) \urcorner$; let the resulting formula be φ_3;

Step 4: Replace each occurrence of $\ulcorner \hat{\alpha}\kappa(\alpha) \cap \hat{\beta}\lambda(\beta) \urcorner$ in φ_3 by $\ulcorner \hat{\alpha}(\kappa(\alpha) \,.\, \lambda(\alpha)) \urcorner$; let the resulting formula be φ_4;

Step 5: Replace each occurrence of $\ulcorner \hat{\alpha}\kappa(\alpha) \cup \hat{\beta}\lambda(\beta) \urcorner$ in φ_4 by $\ulcorner \hat{\alpha}(\kappa(\alpha) \vee \lambda(\alpha)) \urcorner$; let the resulting formula be φ_5;

Step 6: Replace each occurrence of $\ulcorner \hat{\alpha}\kappa(\alpha) \subset \hat{\beta}\lambda(\beta) \urcorner$ in φ_5 by $\ulcorner (\alpha)(\kappa(\alpha) \supset \lambda(\alpha)) \urcorner$; let the resulting formula be φ_6;

Step 7: Replace each occurrence of $\ulcorner \hat{\alpha}\kappa(\alpha) \neq \hat{\beta}\lambda(\beta) \urcorner$ in φ_6 by $\ulcorner {\sim}(\alpha)(\kappa(\alpha) \equiv \lambda(\alpha)) \urcorner$; let the resulting formula be φ_7;

Step 8: Replace each occurrence of $\ulcorner \hat{\alpha}\kappa(\alpha) = \hat{\beta}\lambda(\beta) \urcorner$ in φ_7 by $\ulcorner (\alpha)(\kappa(\alpha) \equiv \lambda(\alpha)) \urcorner$; let the resulting formula be φ'.

As an exercise let us form the monadic associate of the Boolean formula '$\overline{\hat{x}F(x)} \subset (\overline{\hat{x}F(x)} \cup \hat{x}G(x))$'; '$\overline{\hat{x}F(x)} \subset (\overline{\hat{x}F(x)} \cup \hat{x}G(x))$' becomes '$\hat{x}{\sim}F(x) \subset (\hat{x}{\sim}F(x) \cup \hat{x}G(x))$', which becomes in turn '$\hat{x}{\sim}F(x) \subset \hat{x}({\sim}F(x) \vee G(x))$', which becomes in turn '$(x)({\sim}F(x) \supset ({\sim}F(x) \vee G(x)))$'. Let us next form the monadic associate of the Boolean formula '$\hat{x}F(x) \neq \hat{x}G(x) \equiv \hat{x}F(x) = \overline{\hat{x}G(x)}$'; '$\hat{x}F(x) \neq \hat{x}G(x) \equiv \hat{x}F(x) = \overline{\hat{x}G(x)}$' becomes '$\hat{x}F(x) \neq \hat{x}G(x) \equiv \hat{x}F(x) = \hat{x}{\sim}G(x)$', which becomes in turn '${\sim}(x)(F(x) \equiv G(x)) \equiv \hat{x}F(x) = \hat{x}{\sim}G(x)$', which becomes in turn '${\sim}(x)(F(x) \equiv G(x)) \equiv (x)F((x) \equiv {\sim}G(x))$'.

[9]The Boolean algebra of classes is named after George Boole, the founder of modern logic.

It is easily proved that:

(*a*) φ' is a well-formed formula of the monadic quantificational calculus and

(*b*) φ is a theorem of the Boolean algebra of classes if and only if φ' is a theorem of the monadic quantificational calculus.

It can be proved, on the other hand, that a well-formed formula of the monadic quantificational calculus is provable as a theorem if and only if it passes the validity test set forth in chapter two, section 20. We may accordingly conclude that a Boolean class formula φ is provable as a theorem if and only if its monadic associate φ' passes the validity test of section 20. The monadic associate '$(x)(\sim F(x) \supset (\sim F(x) \vee G(x)))$' of the Boolean class formula '$\overline{\hat{x}F(x)} \subset (\overline{\hat{x}F(x)} \cup \hat{x}G(x))$', for instance, passes the validity test of section 20; '$\overline{\hat{x}F(x)} \subset (\overline{\hat{x}F(x)} \cup \hat{x}G(x))$' is therefore provable as a theorem. The monadic associate '$\sim(x)(F(x) \equiv G(x)) \equiv (x)(F(x) \equiv \sim G(x))$' of the Boolean class formula '$\hat{x}G(x) \neq \hat{x}G(x) \equiv \hat{x}F(x) = \overline{\hat{x}G(x)}$', on the other hand, does not pass the validity test of section 20; '$\hat{x}F(x) \neq \hat{x}G(x) \equiv \hat{x}F(x) = \overline{\hat{x}G(x)}$' is therefore not provable as a theorem.

We next include a list of Boolean class formulae provable as theorems; to facilitate the printer's task we shall use the three letters 'A', 'B', and 'C' as shorthands for the three abstracts '$\hat{x}F(x)$', '$\hat{x}G(x)$', and '$\hat{x}H(x)$'.

T350: $A \cup \overline{A} = V$.
T351: $A \cap \overline{A} = \Lambda$.
T352: $A \subset A$.
T353: $A = A$.
T354: $A = A \cap A$.
T355: $A = A \cup A$.
T356: $A = \overline{\overline{A}}$.
T357: $A \cap B \subset A$.
T358: $A \cap B \subset B$.
T359: $A \subset A \cup B$.
T360: $B \subset A \cup B$.
T361: $A \cap B = B \cap A$.
T362: $A \cup B = B \cup A$.
T363: $A = B \equiv B = A$.
T364: $(A \cap B) \cap C = A \cap (B \cap C)$.
T365: $(A \cup B) \cup C = A \cup (B \cup C)$.
T366: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
T367: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
T368: $A \subset B \cap C \equiv (A \subset B \,.\, A \subset C)$.

T369: $A \cup B \subset C \equiv (A \subset C \,.\, B \subset C)$.
T370: $(A \subset B \,.\, B \subset C) \supset A \subset C$.
T371: $(A = B \,.\, B = C) \supset A = C$.
T372: $A \subset B \supset (C \cap A \subset C \cap B)$.
T373: $A \subset B \supset (C \cup A \subset C \cup B)$.
T374: $A \subset B \equiv \overline{B} \subset \overline{A}$.
T375: $A = B \equiv \overline{B} = \overline{A}$.
T376: $A = B \equiv (A \subset B \,.\, B \subset A)$.
T377: $A = B \supset A \subset B$.
T378: $A \subset B \equiv \overline{A} \cup B = \mathrm{V}$.
T379: $A \subset B \equiv A \cap \overline{B} = \Lambda$.
T380: $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
T381: $\overline{A \cup B} = \overline{A} \cap \overline{B}$.
T382: $A \subset B \equiv A = A \cap B$.
T383: $A \subset B \equiv B = A \cup B$.
T384: $\overline{\Lambda} = \mathrm{V}$.
T385: $\overline{\mathrm{V}} = \Lambda$.
T386: $A \cap \mathrm{V} = A$.
T387: $A \cup \Lambda = A$.
T388: $A \cap \Lambda = \Lambda$.
T389: $A \cup \mathrm{V} = \mathrm{V}$.
T390: $A \subset \mathrm{V}$.
T391: $\Lambda \subset A$.

The analogy holding between T350–T383 and the schemata recorded in chapter one will be studied in section 38.

36. N-MEMBERED CLASSES

Our next task is to formalize the notion of an n-membered class. We introduce to that effect the auxiliary notion of a unit class. By *the unit class of* x, in symbols: $\{x\}$, we understand the class whose sole member is x; since x belongs to $\hat{y}(y = y)$ and whatever belongs to $\hat{y}(y = x)$ is identical with x, we may define '$\{x\}$' as '$\hat{y}(y = x)$'.

D22: $\ulcorner\{\alpha\}\urcorner \rightarrow \ulcorner\hat{\beta}(\beta = \alpha)\urcorner$, where β is new.

T400: $(x)(x \in \{x\})$. (I, 300, 331) D22
According to T400 x belongs to $\{x\}$.

T401: $(x)(y)(y \in \{x\} \supset y = x)$.
Proof:
(1): $y \in \{x\} \supset y = x$ (I, 107, 331) D22
(2): $(x)(y)(y \in \{x\} \supset y = x)$ (U, (U, 1))
According to T401 whatever belongs to $\{x\}$ is identical with x.

T402: $(x)(\underset{1}{\mathrm{EL}}y)(y \in \{x\})$.

Proof:

(1): $400 \supset x \in \{x\}$	(202)
$\supset (\mathrm{E}y)(y \in \{x\})$	(217)
(2): $(\mathrm{E}y)(y \in \{x\})$	(400, 1)
(3): $(x)(\underset{1}{\mathrm{EL}}y)(y \in \{x\})$	(U, 2) D10a

T403: $(x)(\underset{1}{\mathrm{EM}}y)(y \in \{x\})$.

Proof:

(1): $(w)(y)(z)((w = y \,.\, z = y) \supset w = z)$	(R, 305)
(2): $(w)(w')(z)((w = w' \,.\, z = w') \supset w = z)$	(R, 1)
(3): $2 \supset (w')(z)((y = w' \,.\, z = w') \supset y = z)$	(200)
$\supset (z)((y = x \,.\, z = x) \supset y = z)$	(200)
(4): $(z)((y = x \,.\, z = x) \supset y = z)$	(2, 3)
(5): $(z)((y \in \{x\} \,.\, z \in \{x\}) \supset y = z)$	(I, 3, 331) D22
(6): $(x)(\underset{1}{\mathrm{EM}}y)(y \in \{x\})$	(I, 323, (U, (U, 5)))

T404: $(x)(\underset{1}{\mathrm{EE}}y)(y \in \{x\})$. (A, 402, 403) D12

According to T402–T404 $\{x\}$ has at least one, at most one, and exactly one member.

T405: $(\underset{1}{\mathrm{EE}}x)(x \in \mathrm{A}) \equiv (\mathrm{E}x)(\mathrm{A} = \{x\})$. (I, 324, 331) D14, D22

According to T405 a class A has exactly one member or, as we shall say, is 1-membered, if and only if A is a unit class.

The sum $\{x\} \cup \{y\}$ of the two unit classes $\{x\}$ and $\{y\}$ is sometimes called a *cardinal couple*. According to T406 $\{x\} \cup \{y\}$ has at most two members:

T406: $(x)(y)(\underset{2}{\mathrm{EM}}z)(z \in \{x\} \cup \{y\})$;

according to T407 $\{x\} \cup \{y\}$, where x and y are distinct, has at least two members:

T407: $(x)(y)(x \neq y \supset (\underset{2}{\mathrm{EL}}z)(z \in \{x\} \cup \{y\}))$;

according to T408 $\{x\} \cup \{y\}$, where x and y are distinct, has exactly two members:

T408: $(x)(y)(x \neq y \supset (\underset{2}{\mathrm{EE}}z)(z \in \{x\} \cup \{y\}))$;

and according to T409 a class A is 2-membered if and only if A is a cardinal whose members are distinct from one another:

T409: $(\underset{2}{\mathrm{E}}\mathrm{E}x)(x \in \mathrm{A}) \equiv (\mathrm{E}x)(\mathrm{E}y)(\mathrm{A} = \{x\} \cup \{y\} \,.\, x \neq y)$.

Similar results hold for cardinal triples and, more generally, for cardinal n-tuples ($n > 2$):

(a) The cardinal n-tuples $\ulcorner\{\alpha_1\} \cup \{\alpha_2\} \cup \ldots \cup \{\alpha_n\}\urcorner$, where $\ulcorner\alpha_1 \neq \alpha_2\urcorner$, $\ulcorner\alpha_1 \neq \alpha_3\urcorner, \ldots, \ulcorner\alpha_1 \neq \alpha_n\urcorner$, $\ulcorner\alpha_2 \neq \alpha_3\urcorner, \ldots, \ulcorner\alpha_2 \neq \alpha_n\urcorner, \ldots,$ $\ulcorner\alpha_{n-1} \neq \alpha_n\urcorner$, have at least, at most, and exactly n members;

(b) A class is n-membered if and only if it is identical with some cardinal n-tuple whose n members are all distinct from one another.

A few extra theorems on unit classes are appended:

T410: $(x)(y)(x \in \{y\} \equiv y \in \{x\})$.
T411: $(x)(y)(\{x\} = \{y\} \equiv x = y)$.
T412: $(x)(x \in \mathrm{A} \equiv \{x\} \subset \mathrm{A})$.
T413: $(x)(y)(x \in \mathrm{A} \cap \overline{\{y\}} \equiv (x \in \mathrm{A} \,.\, x \neq y))$.

According to T413 $A \cap \overline{\{y\}}$ is the class of all the members of A minus y; if y is a member of A, then A will have $n + 1$ members if and only if $A \cap \overline{\{y\}}$ has n members.

T414: $(y)(y \in \mathrm{A} \supset ((\underset{n+1}{\mathrm{E}}\mathrm{E}x)(x \in \mathrm{A}) \equiv (\underset{n}{\mathrm{E}}\mathrm{E}x)(x \in \mathrm{A} \cap \overline{\{y\}})))$.

37. RELATIONS AND TWO-PLACE ABSTRACTS

A monadic predicate is true of one entity at a time; a dyadic predicate, on the other hand, is true of two entities at a time and of two entities in a given order. The dyadic predicate 'is the father of', for instance, is true of the two entities Adam and Cain in the order: Adam, Cain, but false of the two entities Adam and Cain in the reverse order: Cain, Adam. Two entities in a given order make up what we call an *ordinal couple*, as opposed to a mere cardinal couple. We shall thus say that a dyadic predicate is true of ordinal couples. The dyadic predicate 'is the father of', for instance, is true of the ordinal couple: Adam, Cain; of the ordinal couple: Adam, Abel; and so on. The dyadic predicate 'is the son of', on the other hand, is true of the ordinal couple: Cain, Adam; of the ordinal couple: Abel, Adam; and so on.

The various ordinal couples of which a dyadic predicate '—— —— ——' is true may be regarded as forming a relation, the relation of all x's to all y's such that —— x —— y ——, called *the extension of* '—— —— ——'. The various ordinal couples of which the predicate 'is the father of' is true, for instance, may be regarded as forming a relation, the relation of all x's to all y's such that x is the father of y or, more briefly, the relation father of; similarly the various ordinal couples of

which the predicate 'is the son of' is true may be regarded as forming a relation, the relation of all x's to all y's such that x is the son of y or, more briefly, the relation son of. Now it is clear that a given relation will be borne by a given entity w to another entity z if and only if the predicate of which it is the extension is true of the ordinal couple w, z. For instance, the relation father of will be borne by a given entity w to another entity z if and only if the predicate 'is the father of' is true of the ordinal couple w,z; similarly the relation son of will be borne by a given entity w to another entity z if and only if the predicate 'is the son of' is true of the ordinal couple w,z. If we abbreviate 'w bears ... to z' as '$w(\ldots)z$' and 'the relation of all x's to all y's such that $F(x,y)$' as '$\hat{x}\hat{y}F(x,y)$', we may accordingly treat '$w(\hat{x}\hat{y}F(x,y))z$' as a paraphrase of the formula '$F(w,z)$'.

Generalizing upon this remark, we adopt the following definition:

D23: Let $\varphi_{-\gamma\delta}$ be the result of respectively replacing all the bound occurrences of γ and δ in φ by occurrences of any two argument variables foreign to φ; then:

$\ulcorner(\gamma(\hat{\alpha}\hat{\beta}\varphi)\delta)\urcorner \rightarrow \varphi'$, where φ' is like $\varphi_{-\gamma\delta}$ except for containing occurrences of γ and δ wherever $\varphi_{-\gamma\delta}$ respectively contains free occurrences of α and β.[10]

The formulae $\ulcorner\hat{\alpha}\hat{\beta}\varphi\urcorner$ will be called *two-place* or *relation abstracts*; the formulae $\ulcorner\gamma(\hat{\alpha}\hat{\beta}\varphi)\delta\urcorner$, *relation formulae.*[11]

As a predicate dummy, say 'F', serves as a place-holder for predicates, so the abstract '$\hat{x}\hat{y}F(x,y)$' will serve as a place-holder for all the relation names which result from substituting predicates for 'F' in '$\hat{x}\hat{y}F(x,y)$'; it will, for example, serve as a place-holder for the relation names '$\hat{x}\hat{y}$(x is the father of y)', '$\hat{x}\hat{y}$(x is the son of y)', and so on, which respectively result from substituting 'is the father of', 'is the son of', and so on, for 'F' in '$\hat{x}\hat{y}F(x,y)$'.

Definitions D24–D31 introduce eight extra relation formulae into the identity calculus.

(*a*) Two relations $\hat{x}\hat{y}F(x,y)$ and $\hat{w}\hat{z}G(w,z)$ will be said *to be identical*, in symbols:

$$\hat{x}\hat{y}F(x,y) = \hat{w}\hat{z}G(w,z),$$

[10]The simpler definition:

D23: $\ulcorner(\gamma(\hat{\alpha}\hat{\beta}\varphi)\delta)\urcorner \rightarrow \varphi'$, where φ' is like φ except for containing free occurrences of γ and δ wherever φ respectively contains free occurrences of α and β,

fails to account for such formulae as '$w(\hat{x}\hat{y}(w)F(x,y,w))z$', '$w(\hat{x}\hat{y}(z)F(x,y,z))z$', '$w(\hat{x}\hat{y}(w)(z)F(x,y,w,z))z$', and so on, where at least one of γ and δ is bound in φ. D23 will, however, be forthcoming as a metatheorem (MT431).

[11]In set theory ordinal couples may be defined as classes of a sort, and relations defined in turn as classes of ordinal couples.

when whatever bears the first relation to a given entity also bears the second relation to that entity and *vice-versa*, in symbols:

$$(x')(y')(x'(\hat{x}\hat{y}F(x,y))y' \equiv x'(\hat{w}\hat{z}G(w,z))y');$$

the relation brother of, for instance, is identical with the relation male sibling of.

D24: $\ulcorner(\hat{\alpha}\hat{\beta}\varphi = \hat{\gamma}\hat{\delta}\psi)\urcorner \rightarrow \ulcorner(\alpha')(\beta')(\alpha'(\hat{\alpha}\hat{\beta}\varphi)\beta' \equiv \alpha'(\hat{\gamma}\hat{\delta}\psi)\beta')\urcorner$, where α' and β' are new and distinct.

(*b*) Two relations $\hat{x}\hat{y}F(x,y)$ and $\hat{w}\hat{z}G(w,z)$ will be said *to be distinct,* in symbols:

$$\hat{x}\hat{y}F(x,y) \neq \hat{w}\hat{z}G(w,z),$$

when they are not identical, in symbols:

$$\sim(\hat{x}\hat{y}F(x,y) = \hat{w}\hat{z}G(w,z));$$

the relation sibling of, for instance, is distinct from the relation brother of.

D25: $\ulcorner(\hat{\alpha}\hat{\beta}\varphi \neq \hat{\gamma}\hat{\delta}\psi)\urcorner \rightarrow \ulcorner\sim(\hat{\alpha}\hat{\beta}\varphi = \hat{\gamma}\hat{\delta}\psi)\urcorner$.

(*c*) A relation $\hat{x}\hat{y}F(x,y)$ will be said *to be included in* or *to be a subrelation of* another relation $\hat{w}\hat{z}G(w,z)$, in symbols:

$$\hat{x}\hat{y}F(x,y) \subset \hat{w}\hat{z}G(w,z),$$

when whatever bears the first relation to a given entity also bears the second relation to that entity, in symbols:

$$(x')(y')(x'(\hat{x}\hat{y}F(x,y))y' \supset x'(\hat{w}\hat{z}G(w,z))y');$$

the relation husband of, for instance, is included in the relation spouse of.

D26: $\ulcorner(\hat{\alpha}\hat{\beta}\varphi \subset \hat{\gamma}\hat{\delta}\psi)\urcorner \rightarrow \ulcorner(\alpha')(\beta')(\alpha'(\hat{\alpha}\hat{\beta}\varphi)\beta' \supset \alpha'(\hat{\gamma}\hat{\delta}\psi)\beta')\urcorner$, where α' and β' are new and distinct.

(d) By *the sum of two relations* $\hat{x}\hat{y}F(x,y)$ and $\hat{w}\hat{z}G(w,z)$, in symbols:

$$\hat{x}\hat{y}F(x,y) \cup \hat{w}\hat{z}G(w,z),$$

we shall understand the relation of any x' to any y' such that x' bears to y' at least one of the two relations, in symbols:

$$\widehat{x'}\widehat{y'}(x'(\hat{x}\hat{y}F(x,y))y' \vee x'(\hat{w}\hat{z}G(w,z))y');$$

the relation spouse of, for instance, is the sum of the two relations husband of and wife of.

D27: $\ulcorner(\hat{\alpha}\hat{\beta}\varphi \cup \hat{\gamma}\hat{\delta}\psi)\urcorner \rightarrow \ulcorner\widehat{\alpha'}\widehat{\beta'}(\alpha'(\hat{\alpha}\hat{\beta}\varphi)\beta' \vee \alpha'(\hat{\gamma}\hat{\delta}\psi)\beta')\urcorner$, where α' and β' are new and distinct.

(*e*) By *the product of two relations* $\hat{x}\hat{y}F(x,y)$ and $\hat{w}\hat{z}G(w,z)$, in symbols:

$$\hat{x}\hat{y}F(x,y) \cap \hat{w}\hat{z}G(w,z),$$

we shall understand the relation of any x' to any y' such that x' bears to y' both relations, in symbols:

$$\widehat{x'}\widehat{y'}x'(\hat{x}\hat{y}F(x,y))y' \;.\; x'(\hat{w}\hat{z}G(w,z))y');$$

the relation preferred son of, for instance, is the product of the two relations preferred by and son of.

D28: $\ulcorner(\hat{\alpha}\hat{\beta}\varphi \cap \hat{\gamma}\hat{\delta}\psi)\urcorner \rightarrow \ulcorner\widehat{\alpha'}\widehat{\beta'}(\alpha'(\hat{\alpha}\hat{\beta}\varphi)\beta' \;.\; \alpha'(\hat{\gamma}\hat{\delta}\psi)\beta')\urcorner$, where α' and β' are new and distinct.

(*f*) By *the complement of a relation* $\hat{x}\hat{y}F(x,y)$, in symbols:

$$\overline{\hat{x}\hat{y}F(x,y)},$$

we shall understand the relation of any w to any z such that w does not bear $\hat{x}\hat{y}F(x,y)$ to z, in symbols:

$$\hat{w}\hat{z}{\sim}(w(\hat{x}\hat{y}G(x,y))z);$$

the relation indivisible by, for instance, is the complement of the relation divisible by.

D29: $\ulcorner\overline{\hat{\alpha}\hat{\beta}\varphi}\urcorner \rightarrow \ulcorner\hat{\gamma}\hat{\delta}{\sim}(\gamma(\hat{\alpha}\hat{\beta}\varphi)\delta)\urcorner$, where γ and δ are new and distinct.[12]

(*g*) The predicate of which a relation is the extension may, like the predicate 'is identical with and distinct from', be true of no ordinal couple, or, like the predicate 'is the father of', be true of some ordinal couples short of all, or, like the predicate 'is identical with or distinct from', be true of all ordinal couples. By *the null relation* $\dot{\Lambda}$ we shall understand the extension of the predicate 'is identical with and distinct from', by *the universal relation* $\dot{V}$ the extension of the predicate 'is identical with or distinct from'. '$\dot{\Lambda}$' and '$\dot{V}$' are definable in the identity calculus as follows:

D30: '$\dot{\Lambda}$' $\rightarrow$ '$\hat{x}\hat{y}(x = y \;.\; x \neq y)$';
D31: '$\dot{V}$' $\rightarrow$ '$\hat{x}\hat{y}(x = y \vee x \neq y)$'.

The following metatheorems and theorems duplicate MT330–MT346:

MT430: $\ulcorner\gamma(\hat{\alpha}\hat{\beta}\varphi)\delta \equiv \varphi'\urcorner$, where γ and δ are not bound in φ and φ' is like φ except for containing occurrences of γ and δ wherever φ respectively contain free occurrences of α and β.

[12]The six relation signs '=', '≠', '⊂', '∪', '∩', and '—' are easily told from their class analogues by the expressions they govern: relation abstracts as opposed to class abstracts.

MT431: $\ulcorner\gamma(\hat{\alpha}\hat{\beta}\varphi) \equiv \varphi'\urcorner$, where φ' is like φ except for containing free occurrences of γ and δ wherever φ respectively contains free occurrences of α and β.
MT432: $\ulcorner\alpha(\hat{\alpha}\hat{\beta}\varphi)\beta \equiv \varphi\urcorner$.
MT433: $\ulcorner\hat{\alpha}\hat{\beta}\varphi = \hat{\gamma}\hat{\delta}(\gamma(\hat{\alpha}\hat{\beta}\varphi)\delta)\urcorner$, where γ and δ are not free in φ.
MT434: $\ulcorner(\alpha_1)(\alpha_2) \ldots (\alpha_n)(\hat{\beta}\hat{\gamma}\varphi = \widehat{\hat{\delta}\delta'}\psi) \supset (\chi \equiv \omega)\urcorner$, where ω is like χ except for containing $\ulcorner\widehat{\hat{\delta}\delta'}\psi\urcorner$ at some places where χ contains $\ulcorner\hat{\beta}\hat{\gamma}\varphi\urcorner$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all the free argument variables of $\ulcorner\hat{\beta}\hat{\gamma}\varphi\urcorner$ and $\ulcorner\widehat{\hat{\delta}\delta'}\psi\urcorner$
MT435: $\ulcorner\hat{\alpha}\hat{\beta}\varphi = \hat{\gamma}\hat{\delta}\psi \equiv (\alpha)(\beta)(\varphi \equiv \psi')\urcorner$, where α and β do not occur in ψ and ψ' is like ψ except for containing occurrences of α and β wherever ψ respectively contains free occurrences of γ and δ.
MT436: $\ulcorner\hat{\alpha}\hat{\beta}\varphi \subset \hat{\gamma}\hat{\delta}\psi \equiv (\alpha)(\beta)(\varphi \supset \psi')\urcorner$, where α, β, and ψ' are as in MT435.
MT437: $\ulcorner\hat{\alpha}\hat{\beta}\varphi \cup \hat{\gamma}\hat{\delta}\psi = \hat{\alpha}\hat{\beta}(\varphi \vee \psi')\urcorner$, where α, β, and ψ' are as in MT435.
MT438: $\ulcorner\hat{\alpha}\hat{\beta}\varphi \cap \hat{\gamma}\hat{\delta}\psi = \hat{\alpha}\hat{\beta}(\varphi \,.\, \psi')\urcorner$, where α, β, and ψ' are as in MT435.
MT439: $\ulcorner\overline{\hat{\alpha}\hat{\beta}\varphi} = \hat{\alpha}\hat{\beta}{\sim}\varphi\urcorner$.
MT440: $\ulcorner\alpha'(\hat{\alpha}\hat{\beta}\varphi \cup \hat{\gamma}\hat{\delta}\psi)\beta' \equiv (\alpha'(\hat{\alpha}\hat{\beta}\varphi)\beta' \vee \alpha'(\hat{\gamma}\hat{\delta}\psi)\beta')\urcorner$.
MT441: $\ulcorner\alpha'(\hat{\alpha}\hat{\beta}\varphi \cap \hat{\gamma}\hat{\delta}\psi)\beta' \equiv (\alpha'(\hat{\alpha}\hat{\beta}\varphi)\beta' \,.\, \alpha'(\hat{\gamma}\hat{\delta}\psi)\beta')\urcorner$.
MT442: $\ulcorner\gamma(\overline{\hat{\alpha}\hat{\beta}\varphi})\delta \equiv {\sim}(\gamma(\hat{\alpha}\hat{\beta}\varphi)\delta)\urcorner$.
MT443: $\ulcorner\hat{\alpha}\hat{\beta}\varphi = \dot{\Lambda} \equiv {\sim}(\mathrm{E}\alpha)(\mathrm{E}\beta)\varphi\urcorner$.
MT444: $\ulcorner\hat{\alpha}\hat{\beta}\varphi = \dot{\mathrm{V}} \equiv (\alpha)(\beta)\varphi\urcorner$.
T445: ${\sim}(\mathrm{E}x)(\mathrm{E}y)(x \,\dot{\Lambda}\, y)$.
T446: $(x)(y)(x \,\dot{\mathrm{V}}\, y)$.

38. THE BOOLEAN ALGEBRA OF RELATIONS

A *Boolean relation formula* may be defined (in analogy with a Boolean class formula) as a well-formed formula of the identity calculus which contains only:

1. sentential connectives;
2. two-place abstracts $\ulcorner\hat{\alpha}\hat{\beta}\kappa(\alpha,\beta)\urcorner$;
3. Boolean signs, that is, '$=$', '$\neq$', '$\subset$', '$\cup$', '$\cap$', '—', '$\dot{\Lambda}$', and/or '$\dot{\mathrm{V}}$'.

Boolean relation formulae make up a fragment of the elementary calculus of relations known as the *Boolean algebra of relations.*[13]

There exists a mechanical procedure for turning a Boolean relation formula φ into a Boolean class formula φ', called *the class associate of* φ. It consists of two steps, to be taken in any order:

Step 1: Replace each occurrence of a two place abstract $\ulcorner\hat{\alpha}\hat{\beta}\kappa(\alpha,\beta)\urcorner$ by an occurrence of the one-place abstract $\ulcorner\hat{\alpha}\kappa(\alpha)\urcorner$;

[13]The Boolean algebra of relations was first elaborated by the British logician A. De Morgan and the American logician C. S. Peirce.

Step 2: Replace each occurrence of '$\dot{\Lambda}$' and '$\dot{\mathrm{V}}$' by an occurrence of 'Λ' and 'V' respectively.

It is easily shown that φ' is a Boolean class formula and that φ is a theorem of the Boolean algebra of relations if and only if φ' is a theorem of the Boolean algebra of classes. Since we already have a mechanical procedure for deciding whether any Boolean class formula is provable as a theorem or not, we have, by extension, a mechanical procedure for deciding whether any Boolean relation formula is provable as a theorem or not.

All the Boolean relation formulae whose class associates are listed in section 35 are provable as theorems; the reader is asked to record them on his own as T450–T491. To simplify his task, he may use the letter 'P' as an abbreviation for the two-place abstract '$\hat{x}\hat{y}F(x,y)$', the letter 'Q' as an abbreviation for the two-place abstract '$\hat{x}\hat{y}G(x,y)$', and the letter 'R' as an abbreviation for the two-place abstract '$\hat{x}\hat{y}H(x,y)$'; we shall follow the same convention when recording theorems of our own below.

We have met another Boolean algebra before: the sentential calculus. Since

'$\equiv$' behaves like '$=$',
'$\not\equiv$' " " '$\neq$',
'$\supset$' " " '$\subset$',
'$\vee$' " " '$\cup$',
'$\cdot$' " " '$\cap$',
'$\sim$' " " '$-$',
$\ulcorner\sim(\varphi \supset \varphi)\urcorner$ " " 'Λ',

and

$\ulcorner\varphi \supset \varphi\urcorner$ " " 'V',

the sentential calculus may indeed be identified as a *Boolean algebra of statements*.

We remarked above that the theorems of the Boolean algebra of classes are analogous to the theorems of the sentential calculus; '$\hat{x}F(x) \cup \hat{x}G(x) = \hat{x}G(x) \cup \hat{x}F(x)$', for instance, is analogous to '$(p \vee q) \equiv (q \vee p)$'. This observation is formalized in the following metatheorem:

MT495: Let φ be a Boolean class formula, let $\ulcorner\widehat{\alpha_1}\kappa_1(\alpha_1)\urcorner$, $\ulcorner\widehat{\alpha_2}\kappa_2(\alpha_2)\urcorner, \ldots,$ $\ulcorner\widehat{\alpha_n}\kappa_n(\alpha_n)\urcorner$ respectively be the first, the second, . . . , the n-th one-place abstracts occurring in φ, and let φ' be the result of:

1. replacing each occurrence of $\ulcorner\widehat{\alpha_i}\kappa_i(\alpha_i)\urcorner$ in φ by the i-th sentential dummy from the list:

'p', 'q', 'r', 's', 'p'', 'q'', 'r'', 's'', . . . ;

2. replacing each occurrence of '$=$', '$\neq$', '$\subset$', '$\cup$', '$\cap$', '$-$', 'Λ', and 'V' in φ by an occurrence of '$\equiv$', '$\not\equiv$', '$\supset$', '$\vee$', '$.$', '$\sim$', '$\sim(p \supset p)$' and '$p \supset p$'.

If φ is a theorem of the Boolean algebra of classes, then φ' is a theorem of the sentential calculus.

A similar metatheorem holds, of course, for Boolean relation formulae.

The converse of MT495 is false. Let, for instance, φ be '$\hat{x}F(x) \neq \hat{x}G(x) \equiv \hat{x}F(x) = \overline{\hat{x}G(x)}$'; φ' will then be '$(p \not\equiv q) \equiv (p \equiv \sim q)$'. '$\hat{x}F(x) \neq \hat{x}G(x) \equiv \hat{x}F(x) = \overline{\hat{x}G(x)}$', we know, is not a theorem of the Boolean algebra of classes; yet '$(p \not\equiv q) \equiv (p \equiv \sim q)$' is a theorem of the sentential calculus. Some theorems of the sentential calculus have therefore no class (or relation) analogue. This anomaly is easily explained.

The formula:

$$(p \equiv (q \supset q)) \vee (p \equiv \sim(q \supset q)),$$

according to which any given formula is equivalent to what we may call *the universal formula* '$q \supset q$' or equivalent to what we may call *the null formula* '$\sim(q \supset q)$', is a theorem of the sentential calculus. The corresponding formula:

$$A = V \vee A = \Lambda,$$

on the other hand, is not a theorem of the Boolean algebra of classes; a class may indeed be distinct from the universal class without being automatically identical with the null class. The Boolean algebra of statements thus reckons only two elements, a universal formula and a null one, whereas the Boolean algebra may reckon besides V and Λ an indefinite number of elements. The theorems of the sentential calculus which have no class (or relation) analogue are simply formulae which hold in a *two-element* Boolean algebra, but fail in an *n-element* ($n > 2$) one.

The three Boolean algebras of classes, relations, and statements are generalizations of the familiar algebra of real numbers,

the Boolean predicate '$=$' behaving like the numerical predicate '$=$',
" " " '$\neq$' " " " " " '$\neq$',
" " " '$\subset$' " " " " " '$\leq$',
" " operator '$\cup$' " " " " operator '$+$',
" " " '$\cap$' " " " " " '$\times$',
" " " '$-$' " " " " " '$1-$',
" " abstract 'Λ' " " " numeral '0',
" " " 'V' " " " " '1'.[14]

[14] '$\cup$' also behaves like the operator 'max', where $\max(x,y)$ is the larger of the two numbers x and y, and '$\cap$' also behaves like the operator 'min', where $\min(x,y)$ is the smaller of the two numbers x and y.

The analogy between 'Λ' and '0' is borne out by the two formulae:

$$A \cup \Lambda = A \qquad \text{and} \qquad x + 0 = x;$$

the analogy between 'V' and '1' is borne out by the two formulae:

$$A \cap \mathrm{V} = A \qquad \text{and} \qquad x \times 1 = x.$$

Boolean algebras differ from the algebra of real numbers in two main respects:

1. They admit no coefficients; so whereas $x + x$ equals $2x$, $A \cup A$ equals A;
2. They admit no powers; so whereas $x \times x$ equals x^2, $A \cap A$ equals A.

39. CONVERSES, RELATIVE PRODUCTS, AND IMAGES

The converse of a relation P, in symbols: $\breve{P}$, is the relation which a given entity x bears to another entity y if and only if y bears P to x. The converse of the relation smaller than, for instance, is the relation greater than; the converse of the relation husband of, the relation wife of.

D32: $\ulcorner \overset{\smile}{\hat{\alpha}\hat{\beta}\varphi} \urcorner \rightarrow \ulcorner \hat{\beta}\hat{\alpha}\varphi \urcorner$.

MT500: $\ulcorner \gamma(\overset{\smile}{\hat{\alpha}\hat{\beta}\varphi})\delta \equiv \delta(\hat{\alpha}\hat{\beta}\varphi)\gamma \urcorner$.

T501: $P = \breve{\breve{P}}$.

According to T501 two converse signs cancel like two denial or two complement signs.

T502: $P \subset Q \equiv \breve{P} \subset \breve{Q}$.

For instance, the relation smaller than is included in the relation smaller than or equal to if and only if the relation larger than is included in the relation larger than or equal to.

T503: $P = Q \equiv \breve{P} = \breve{Q}$. (I, 500, 242) D24

T504: $\overset{\smile}{P \cap Q} = \breve{P} \cap \breve{Q}$.

T505: $\overset{\smile}{P \cup Q} = \breve{P} \cup \breve{Q}$.

T506: $\breve{\bar{P}} = \bar{\breve{P}}$.

The relative product of two relations P and Q, in symbols: $P \mid Q$, is the relation which a given entity x bears to another entity y if and only if x bears P to some entity which bears Q to y. The relative product of the

two relations mother of and father of, for instance, is the relation paternal grandmother of; the relative product of the two relations father of and mother of is the relation maternal grandfather of.

D33: $\ulcorner(\hat{\alpha}\hat{\beta}\varphi \mid \hat{\gamma}\hat{\delta}\psi)\urcorner \rightarrow \ulcorner\widehat{\alpha'}\widehat{\beta'}(\mathrm{E}\gamma)(\alpha'(\hat{\alpha}\hat{\beta}\varphi)\gamma \,.\, \gamma(\hat{\gamma}\hat{\delta}\psi)\beta')\urcorner$, where α', β', and γ' are new and distinct.[15]

MT510: $\ulcorner\alpha'(\hat{\alpha}\hat{\beta}\varphi \mid \hat{\gamma}\hat{\delta}\psi)\beta' \equiv (\mathrm{E}\gamma')(\alpha'(\hat{\alpha}\hat{\beta}\varphi)\gamma' \,.\, \gamma'(\hat{\gamma}\hat{\delta}\psi)\beta')\urcorner$, where γ' is not free in φ nor in ψ.

T511: $P \subset Q \supset (P \mid R) \subset (Q \mid R)$.

For instance, the relation father of being included in the relation parent of, the relative product maternal grandfather of is included in the relative product maternal grandparent of.

T512: $P \subset Q \supset (R \mid P) \subset (R \mid Q)$.

For instance, the relation father of being included in the relation parent of, the relative product paternal grandmother of is included in the relative product grandmother of.

T513: $(P \mid Q) \mid R = P \mid (Q \mid R)$.
T514: $P \mid (Q \cup R) = (P \mid Q) \cup (P \mid R)$.
T515: $(P \cup Q) \mid R = (P \mid R) \cup (Q \mid R)$.

The analogous laws for '$\cap$' hold as inclusions only:

T516: $P \mid (Q \cap R) \subset (P \mid Q) \cap (P \mid R)$.
T517: $(P \cap Q) \mid R \subset (P \mid R) \cap (Q \mid R)$.

Though associative and partly distributive, '|' is not commutative; the relation maternal grandfather of, for instance, is distinct from the relation paternal grandmother of. Note, however:

T518: $\overset{\smile}{P \mid Q} = \check{Q} \mid \check{P}$.

Proof:

(1): $x(P \mid Q)y \equiv (\mathrm{E}z)(zQy \,.\, xPz)$ (I, 137, 510)

(2): $y(\overset{\smile}{P \mid Q})x \equiv (\mathrm{E}z)(y\check{Q}z \,.\, z\check{P}x)$ (I, 500, 1 thrice)

(3): $y(\overset{\smile}{P \mid Q})x \equiv y(\check{Q} \mid \check{P})x$ (I, 510, 2)

(4): $\overset{\smile}{P \mid Q} = \check{Q} \mid \check{P}$ (U, (U, 3)) D24

[15]The relation sign '|' is easily told from the sentential connective '|' (D6) by the expressions it governs: relation abstracts as opposed to well-formed formulae.

Let, for instance, P be the relation teacher of and Q be the relation child of; then by T518 the converse of the relation teacher of child of is the relation parent of pupil of.

The image of a class A with respect to a relation P, in symbols: $P``A$, is the class of all the entities which bear P to some member or other of A. The image of the class of printers with respect to the relation father of, for instance, is the class of fathers of painters; the image of the class of fathers with respect to the relation painter of is the class of painters of fathers.

D34: $\ulcorner(\hat{\alpha}\hat{\beta}\varphi``\hat{\gamma}\psi)\urcorner \rightarrow \ulcorner\hat{\delta}(\mathrm{E}\delta')(\delta' \in \hat{\gamma}\psi \,.\, \delta(\hat{\alpha}\hat{\beta}\varphi)\delta')\urcorner$, where δ and δ' are new and distinct.

MT520: $\ulcorner\delta \in (\hat{\alpha}\hat{\beta}\varphi``\hat{\gamma}\psi) \equiv (\mathrm{E}\delta')(\delta' \in \hat{\gamma}\psi) \,.\, \delta(\hat{\alpha}\hat{\beta}\varphi)\delta')\urcorner$, where δ' is not free in φ.

T521: $(P \cup Q)``A = (P``A) \cup (Q``A)$.

For instance, the brothers or sisters of painters are the brothers of painters or the sisters of painters.

T522: $P``(A \cup B) = (P``A) \cup (P``B)$.

For instance, the wives of pianists or violinists are the wives of pianists or the wives of violinists.

The analogous laws for '$\cap$' hold as inclusions only:

T523: $(P \cap Q)``A \subset (P``A) \cap (Q``A)$.
T524: $P``(A \cap B) \subset (P``A) \cap (P``B)$.
T525: $P \subset Q \supset (P``A) \subset (Q``A)$.
T526: $P = Q \supset (P``A) = (Q``A)$.
T527: $A \subset B \supset (P``A) \subset (P``B)$.
T528: $A = B \supset (P``A) = (P``B)$.

Note the following:

T529: $(P \mid Q)``A = P``(Q``A)$.

Proof:

(1):	$x \in (P \mid Q)``A \equiv (\mathrm{E}y)(y \in A \,.\, (\mathrm{E}z)(xPz \,.\, zQy))$	(I, 520, 510)
(2):	$x \in (P \mid Q)``A \equiv (\mathrm{E}y)(y \in A \,.\, (\mathrm{E}z)(zQy \,.\, xPz))$	(I, 137, 1)
(3):	$x \in (P \mid Q)``A \equiv (\mathrm{E}y)(\mathrm{E}z)(y \in A \,.\, (zQy \,.\, xPz)$	(I, 229, 2)
(4):	$x \in (P \mid Q)``A \equiv (\mathrm{E}y)(\mathrm{E}z)((y \in A \,.\, zQy) \,.\, xPz)$	(I, 139, 3)
(5):	$x \in (P \mid Q)``A \equiv (\mathrm{E}z)(\mathrm{E}y)((y \in A \,.\, zQy) \,.\, xPz)$	(I, 243, 4)
(6):	$x \in (P \mid Q)``A \equiv (\mathrm{E}z)((\mathrm{E}y)(y \in A \,.\, zQy) \,.\, xPz)$	(I, 229, 5)
(7):	$x \in (P \mid Q)``A \equiv (\mathrm{E}z)(z \in Q``A \,.\, xPz)$	(I, 520, 6)
(8):	$x \in (P \mid Q)``A \equiv x \in P``(Q``A)$	(I, 520, 7)
(9):	$(P \mid Q)``A = P``(Q``A)$	(U, 8) D14

For instance, the aunts of pianists are the sisters of the parents of pianists.

The following four images are of special interest:

(*a*) $R``\{y\}$ is identical with $\hat{x}(\mathrm{E}w)(w \epsilon \{y\} \,.\, xRw)$, which is identical in turn with $\hat{x}(\mathrm{E}w)(w = y \,.\, xRw)$, which is identical in turn with $\hat{x}(xRy)$; $R``\{y\}$ is thus the class of all the entities which bear R to y; if R is the relation parent of and y is Napoleon, then $R``\{y\}$ is the class of Napoleon's parents.

(*b*) $\breve{R}``\{x\}$ is identical with $\hat{y}(\mathrm{E}w)(w \epsilon \{x\} \,.\, y\breve{R}w)$, which is identical in turn with $\hat{y}(\mathrm{E}w)(w = x \,.\, y\breve{R}w)$, which is identical in turn with $\hat{y}(y\breve{R}x)$, which is identical in turn with $\hat{y}(xRy)$; $\breve{R}``\{x\}$ is thus the class of all the entities to which x bears R; if R is the relation parent of and x is Napoleon, then $\breve{R}``\{x\}$ is the class of Napoleon's children.

(*c*) $R``\mathrm{V}$ is identical with $\hat{x}(\mathrm{E}y)(y \epsilon \mathrm{V} \,.\, xRy)$, which is identical in turn with $\hat{x}(\mathrm{E}y)(xRy)$; $R``\mathrm{V}$ is thus the class of all the entities which bear R to something or other; if R is the relation parent of, then $R``\mathrm{V}$ is the class of all parents.

(*d*) $\breve{R}``\mathrm{V}$ is identical with $\hat{x}(\mathrm{E}y)(y \epsilon \mathrm{V} \,.\, x\breve{R}y)$, which is identical in turn with $\hat{x}(\mathrm{E}y)(x\breve{R}y)$, which is identical in turn with $\hat{x}(\mathrm{E}y)(yRx)$; $\breve{R}``\mathrm{V}$ is thus the class of all the entities to which something or other bears R; if R is the relation parent of, then $\breve{R}``\mathrm{V}$ is the class of all children.

In the literature $R``\mathrm{V}$ is usually called *the domain of R*, $\breve{R}``\mathrm{V}$ *the converse domain of R*, and $R``\mathrm{V} \cup \breve{R}``\mathrm{V}$ *the field of R*.

40. PROPERTIES OF RELATIONS

Relations may be grouped under several headings:

1. *Reflexivity.*

(*a*) A relation R is said to be *reflexive in a class A* when

$$(x)(x \epsilon A \supset xRx);$$

the relation as old as, for instance, is reflexive in the class of human beings.

(*b*) A relation R is said to be *irreflexive in a class A* when

$$(x)(x \epsilon A \supset \sim(xRx));$$

the relation father of, for instance, is irreflexive in the class of human beings.

(*c*) A relation R is said to be *non-reflexive in a class A* when it is

neither reflexive in A nor irreflexive in A; the relation confident in, for instance, is non-reflexive in the class of human beings.[16]

A relation reflexive, irreflexive, or non-reflexive in the universal class V is said to be *reflexive, irreflexive,* or *non-reflexive.*

2. *Symmetry.*

(*a*) A relation R is said to be *symmetrical in a class A* when

$$(x)(y)((x \epsilon A \,.\, y \epsilon A) \supset (xRy \supset yRx));$$

the relation sibling of, for instance, is symmetrical in the class of human beings.

(*b*) A relation R is said to be *asymmetrical in a class A* when

$$(x)(y)((x \epsilon A \,.\, y \epsilon A) \supset (xRy \supset \sim(yRx)));$$

the relation father of, for instance, is asymmetrical in the class of human beings.

(*c*) A relation R is said to be *non-symmetrical in a class A* if it is neither symmetrical nor asymmetrical in A; the relation admirer of, for instance, is non-symmetrical in a class of human beings.

A relation symmetrical, asymmetrical, or non-symmetrical in the universal class V is said to be *symmetrical, asymmetrical,* or *non-symmetrical.*[16a]

3. *Transitivity.*

(*a*) A relation R is said to be *transitive in a class A* when

$$(x)(y)(z)(((x \epsilon A \,.\, y \epsilon A) \,.\, z \epsilon A) \supset ((xRy \,.\, yRz) \supset xRz));$$

the relation older than, for instance, is transitive in the class of human beings.

(*b*) A relation R is said to be *intransitive in a class A* when

$$(x)(y)(z)(((x \epsilon A \,.\, y \epsilon A) \,.\, z \epsilon A) \supset ((xRy \,.\, yRz) \supset \sim(xRz)));$$

the relation father of, for instance, is intransitive in the class of human beings.

(*c*) A relation R is said to be *non-transitive in a class A* if it is neither transitive nor intransitive in A; the relation friend of, for instance, is non-transitive in the class of human beings.

[16]The phrase 'non-reflexive' is sometimes used to designate a relation which is not reflexive and, hence, may be either irreflexive or non-reflexive in our own sense of the word 'non-reflexive'; the same remark applies to the phrases 'non-symmetrical' and 'non-transitive'.

[16a]The word 'commutative' is sometimes used in place of the word 'symmetrical'; see, for instance, pages 17 and 163.

A relation transitive, intransitive, or non-transitive in the universal class V is said to be *transitive, intransitive,* or *non-transitive.*

A few sample theorems are listed. According to T550 if a relation R is reflexive, then its complement $\bar{R}$ is irreflexive:

T550: $(x)(xRx) \supset (x){\sim}(x\bar{R}x)$;

according to T551 if a relation R is symmetrical, then its complement $\bar{R}$ is also symmetrical:

T551: $(x)(y)(xRy \supset yRx) \supset (x)(y)(x\bar{R}y \supset y\bar{R}x)$;

according to T552 if a relation R is asymmetrical, then it is also irreflexive:

T552: $(x)(y)(xRy \supset {\sim}(yRx)) \supset (x){\sim}(xRx)$.

Proof:

$$\begin{array}{lll}
(1)\colon (xRx \supset {\sim}(xRx)) \supset {\sim}(xRx) & & (100)\quad \text{D2} \\
(2)\colon (x)(y)(xRy \supset {\sim}(yRx)) & \supset (y)(xRy \supset {\sim}(yRx)) & (202) \\
 & \supset (xRx \supset {\sim}(xRx)) & (200) \\
 & \supset {\sim}(xRx) & (1) \\
(3)\colon (x)2 \equiv ((x)(y)(xRy \supset {\sim}(yRx)) \supset (x){\sim}(xRx)) & & (234) \\
(4)\colon (x)(y)(xRy \supset {\sim}(yRx)) \supset (x){\sim}(xRx) & & (\text{I}, 3, (\text{U}, 2))
\end{array}$$

According to T553 a relation R is symmetrical if and only if it is included in its own converse:

T553: $(x)(y)(xRy \supset yRx) \equiv R \subset \breve{R}$;

according to T554 a relation R is asymmetrical if and only if it is included in the complement of its own converse:

T554: $(x)(y)(xRy \supset {\sim}(yRx)) \equiv R \subset \bar{\breve{R}}$;

according to T555 a relation R is transitive if and only if the relative product $R \mid R$ is included in R:

T555: $(x)(y)(z)((xRy \,.\, yRz) \supset xRz) \equiv R \mid R \subset R$;

according to T556 a relation R is intransitive if and only if the relative product $R \mid R$ is included in the complement of R:

T556: $(x)(y)(z)((xRy \,.\, yRz) \supset {\sim}(xRz)) \equiv R \mid R \subset \bar{R}$;

and so on.

41. OF FUNCTIONS

Relations may be grouped into three further classes: many-one, one-many, and one-one relations.

(*a*) A relation R is said to be *many-one in a class A* when

$$(x)(x \in \mathrm{A} \supset \underset{1}{(\mathrm{EE}y)}(x\ R\ y));$$[17]

the relation square root of, for instance, is many-one in the class of real numbers since every real number x is the square root of exactly one y.

(*b*) A relation R is said to be *one-many in a class A* when

$$(y)(y \in \mathrm{A} \supset \underset{1}{(\mathrm{EE}x)}(x\ R\ y));$$

the relation square of, for instance, is one-many in the class of real numbers since every real number y has exactly one square x.

(*c*) A relation R is said to be *one-one in a class A* when it is both many-one and one-many in A; the relation double of, for instance, is one-one in the class of real numbers since every real number x is the double of exactly one y and every real number y has exactly one double x.

According to T560 a relation R is many-one in a class A if and only if $\breve{R}$ is one-many in A:

T560: $(x)(x \in A \supset \underset{1}{(\mathrm{EE}y)}(x\ R\ y)) \equiv (y)(y \in A \supset \underset{1}{(\mathrm{EE}x)}(x\ \breve{R}\ y))$.

Proof:

(1): $(x)(x \in A \supset \underset{1}{(\mathrm{EE}y)}(x\ R\ y)) \equiv (x)(x \in A \supset (\mathrm{E}y)(z)(z\ \breve{R}\ x \equiv z = y))$ (I, 324, 500)

(2): $(x)(x \in A \supset \underset{1}{(\mathrm{EE}y)}(x\ R\ y)) \equiv (x)(x \in A \supset (\mathrm{E}w)(z)(z\ \breve{R}\ x \equiv z = w))$ (R, 1)

(3): $(x)(x \in A \supset \underset{1}{(\mathrm{EE}y)}(x\ R\ y)) \equiv (y)(y \in A \supset (\mathrm{E}w)(z)(z\ \breve{R}\ y \equiv z = w))$ (R, 2)

(4): $(x)(x \in A \supset \underset{1}{(\mathrm{EE}y)}(x\ R\ y)) \equiv (y)(y \in A \supset \underset{1}{(\mathrm{EE}x)}(x\ \breve{R}\ y))$ (I, (R, 3), 324)

According to T561 a relation R is one-many or one-one in a class A if and only if R is one-many in A:

T561: $((y)(y \in A \supset \underset{1}{(\mathrm{EE}x)}(x\ R\ y)) \vee ((x)(x \in A \supset \underset{1}{(\mathrm{EE}y)}(x\ R\ y)) \,.\, (y)(y \in A \supset \underset{1}{(\mathrm{EE}x)}(x\ R\ y)))) \equiv (y)(y \in A \supset \underset{1}{(\mathrm{EE}x)}(x\ R\ y))$.

A relation R which is one-many or one-one in a class A is said to be a *function in A*; the relations square of and double of, for instance, are functions in the class of real numbers. In view of T561 the phrase 'R is a function in A', in symbols: 'Func(R,A)', may be defined as '$(y)(y \in A \supset \underset{1}{(\mathrm{EE}x)}(xRy))$'.

[17]The weaker *definiens* '$(x)(x \in A \supset \underset{1}{(\mathrm{EM}y)}(x\ \mathrm{R}\ y))$' is frequently used; the same remark applies to (b).

D35: $\ulcorner(\mathrm{Func}\,(\hat{\alpha}\hat{\beta}\varphi,\hat{y}\psi))\urcorner \rightarrow \ulcorner(\delta)(\delta \in \hat{y}\psi \supset (\underset{1}{\mathrm{EE}}\delta')(\delta'(\hat{\alpha}\hat{\beta}\varphi)\delta))\urcorner$, where δ and δ' are new and distinct.

T562: $\mathrm{Func}(R,A) \equiv (y)(y \in A \supset (\underset{1}{\mathrm{EE}}x)(xRy))$.

A function in the universal class V is said to be a *function*:

D36: $\ulcorner(\mathrm{Func}(\hat{\alpha}\hat{\beta}\varphi))\urcorner \rightarrow \ulcorner(\mathrm{Func}(\hat{\alpha}\hat{\beta}\varphi,\mathrm{V}))\urcorner$.

The relation $\hat{x}\hat{y}(x = y)$, for instance, is a function in V and hence a function.

T563: $\mathrm{Func}\,(R) \equiv (y)(\underset{1}{\mathrm{EE}}x)(x\,R\,y)$.

The sum of the various classes in which a given relation R is a function is sometimes called *the range of functionality of R*, in symbols: Range_R; it may be defined as $\hat{y}(\underset{1}{\mathrm{EE}}x)(xRy)$.

D37: $\ulcorner\mathrm{Range}_{\hat{\alpha}\hat{\beta}\varphi}\urcorner \rightarrow \ulcorner\hat{\gamma}(\underset{1}{\mathrm{EE}}\delta)(\delta(\hat{\alpha}\hat{\beta}\varphi)\gamma)\urcorner$, where γ and δ are new and distinct.

The range of functionality of the relation spouse of, for instance, is the class of all married human beings; the range of functionality of the relation husband of, the class of all married women.

T570: $\mathrm{Range}_R \subset \breve{R}``\mathrm{V}$.

According to T570 the range of functionality of a relation R is included in the converse domain of R.

T571: $\mathrm{Func}(R) \supset \mathrm{Range}_R = \breve{R}``\mathrm{V}$.

According to T571 the range of functionality of a function R is identical with its converse domain.

T572: $\mathrm{Func}(R) \equiv \mathrm{Range}_R = \mathrm{V}$.

According to T572 the range of functionality of a function R is, as expected, the universal class V.

The relations we have studied here are so-called *dyadic relations* like the relation smaller than; *triadic relations* like the relation betweenness and, more generally, *n-adic* ($n > 2$) *relations* could easily be defined in analogy with D33. Similarly, the functions we have studied here are so-called *one-argument functions* like the functions square of, double of, and so on; *two-argument functions* like the functions plus, times, and so on, and, more generally, *n-argument* ($n > 1$) functions could easily be defined in analogy with D35.[18]

[18] *N*-argument functions are also called *n-ary operations*, and the two words 'functor' and 'operator' used interchangeably to designate a function.

CHAPTER FIVE

Sample Syntax

*42. FOUR SYNTACTICAL CONCEPTS

We shall devote this chapter to four cardinal concepts of syntax: the concepts of consistency, completeness, decidability, and independence.

1. *Consistency.* Calculi may be of two different types: some, called here *calculi of type A*, include only statements among their well-formed formulae; others, called here *calculi of type B*, include schemata and, possibly, statements among their well-formed formulae.

Let us first consider a calculus C of type A; let us assume that certain statements of C have been appointed as axioms of C and certain metalogical rules appointed as rules of deduction in C, and hence that some statements of C are provable as theorems of C; let us also assume that the statements of C have been properly interpreted and hence fall into two classes: true statements of C and false statements of C. It follows from the meaning of '$\sim$' that a statement φ is true if and only if its denial $\ulcorner\sim\varphi\urcorner$ is false and that a statement φ is false if and only if its denial $\ulcorner\sim\varphi\urcorner$ is true. Of any two statements φ and $\ulcorner\sim\varphi\urcorner$ of C at most one is therefore true and hence at most one is wanted as a theorem of C.

Let us next consider a calculus C of type B and assume again that certain formulae of C have been appointed as axioms of C and certain metalogical rules appointed as rules of deduction in C. The schemata of C, when properly interpreted, will fall into three classes: valid, indeterminate, and contravalid schemata of C. It follows from the meaning of '$\sim$' that a schema φ is valid if and only if $\ulcorner\sim\varphi\urcorner$ is contravalid, that a schema φ is contravalid if and only if $\ulcorner\sim\varphi\urcorner$ is valid, and that a schema φ is indeterminate if and only if $\ulcorner\sim\varphi\urcorner$ is indeterminate. Of any two schemata φ and $\ulcorner\sim\varphi\urcorner$ of C at most one is therefore valid and hence at most one is wanted as a theorem of C.

We shall say that a calculus C of either type is *consistent relatively to* '$\sim$' or, more briefly, *relatively consistent,* if of any two well-formed for-

mulae φ and $\ulcorner\sim\varphi\urcorner$ of C at most one is a theorem of C.[1] The sentential calculus, the quantificational calculus, and the identity calculus, for instance, are relatively consistent.

The present concept is obviously applicable only to calculi which contain a denial sign '$\sim$'. Another consistency concept may be defined which is applicable to all calculi, the concept of absolute consistency. We shall say that a calculus C is *absolutely consistent* if there is at least one well-formed formula φ of C which is not a theorem of C.

It is easily seen that any calculus C which includes the formulae:

$$\ulcorner\psi \supset (\sim\psi \supset \varphi)\urcorner,$$

among its theorems and the rule of Detachment among its rules of deduction, is relatively consistent if and only if absolutely consistent. For let C be relatively inconsistent and let φ be any well-formed formula of C; if a formula ψ and its denial $\ulcorner\sim\psi\urcorner$ are both available as theorems of C, φ will by $\ulcorner\psi \supset (\sim\psi \supset \varphi)\urcorner$ and two applications of R1 be a theorem of C, and C will be relatively inconsistent. Let C, on the other hand, be absolutely inconsistent; then, any well-formed formula of C being a theorem of C, any two well-formed formulae φ and $\ulcorner\sim\varphi\urcorner$ of C will be theorems of C, and C will be relatively inconsistent.[2] The sentential calculus, the quantificational calculus, and the identity calculus, among others, are absolutely consistent.

The two concepts of relative and absolute consistency were framed with a semantical purpose in mind; being defined, however, in terms of a syntactical concept, the concept of theoremhood, they are themselves syntactical concepts.

2. *Completeness.* Let us take again a calculus C of type A; since at least one of any two formulae φ and $\ulcorner\sim\varphi\urcorner$ of C is true, then at least one of any two formulae φ and $\ulcorner\sim\varphi\urcorner$ of C is wanted as a theorem of C. We might accordingly consider defining the predicate 'is relatively complete' as follows:

A calculus C is said to be *complete relatively to* '$\sim$' or, more briefly, *relatively complete,* if of any two well-formed formulae φ and $\ulcorner\sim\varphi\urcorner$ of C at least one is a theorem of C (1).

[1]The phrase 'C is relatively consistent' has also been used to mean: 'There is a calculus C' such that if C' is consistent, then C also is'. We shall not study here this second concept of relative consistency.

[2]In a three-valued sentential calculus the formulae:

$$\ulcorner\psi \supset (\sim\psi \supset \varphi)\urcorner,$$

are not provable as theorems, and hence the two concepts of relative and absolute consistency no longer coincide.

The definition, however, is not applicable to calculi of type B. If φ is valid or contravalid, then at least one of the two formulae φ and $\ulcorner \sim\varphi \urcorner$ is wanted as a theorem of C; if, on the other hand, φ is indeterminate, then so is $\ulcorner \sim\varphi \urcorner$, and neither formula is wanted as a theorem of C. A weaker definition is therefore needed to suit calculi of type B as well as calculi of type A. The following has been suggested and is adopted here, namely:

A calculus C is said to be *relatively complete* if C' is relatively inconsistent where C' is like C except for containing as an extra axiom any well-formed formula of C which is not a theorem of C (2).

It is clear that if a calculus C is relatively complete by definition (1), then C is relatively complete by definition (2). For let one of any two formulae φ and $\ulcorner \sim\varphi \urcorner$ of C be a theorem of C and, hence, of C'; then any non-theorem of C, when appointed as an axiom of C', will be the denial of a theorem of C', and C' will be relatively inconsistent. It can also be shown of most calculi C of type A that if C is relatively complete by definition (2), then C is relatively complete by definition (1).[3] A calculus C of type B may, however, be relatively complete by definition (2) without being relatively complete by definition (1); such is the case with

[3]Any calculus C which includes the formulae:

$$\ulcorner(\varphi \supset (\psi \,.\, \sim\psi)) \supset \sim\varphi\urcorner,$$

among its theorems and the rule of Detachment among its rules of deduction, and allows the deduction of

$$\ulcorner\varphi \supset (\psi \,.\, \sim\psi)\urcorner$$

from

$$\ulcorner\varphi \vdash \psi \,.\, \sim\psi\urcorner$$

for any two φ and ψ, is relatively complete by definition (1), if it is relatively complete by definition (2). Note indeed that if, as assumed, C' is relatively inconsistent, then $\ulcorner\psi \,.\, \sim\psi\urcorner$ must be derivable in C from φ, where φ is any non-theorem of C. If, on the other hand,

$$\ulcorner\varphi \supset (\psi \,.\, \sim\psi)\urcorner$$

is deducible in C from

$$\varphi \vdash \ulcorner\psi \,.\, \sim\psi\urcorner,$$

and

$$\ulcorner(\varphi \supset (\psi \,.\, \sim\psi)) \supset \sim\varphi\urcorner$$

is a theorem of C, then $\ulcorner \sim\varphi \urcorner$ is a theorem of C and, hence, of any two formulae φ and $\ulcorner \sim\varphi \urcorner$ at least one is a theorem of C. Most calculi of type A allow the deduction of

$$\ulcorner\varphi \supset (\psi \,.\, \sim\psi)\urcorner$$

from

$$\varphi \vdash \ulcorner\psi \,.\, \sim\psi\urcorner.$$

Few calculi of type B, however, allow that deduction, and those which do are often not relatively complete by definition (2). Of the two versions of the sentential calculus, for instance, version I does not allow the deduction in question (cf. page 130n), and version II, which does, is not relatively complete by definition (2).

version I of the sentential calculus in which neither one of the two formulae 'p' and '$\sim p$' is provable as a theorem and in which neither one of the same two formulae can be appointed as an extra axiom without the resulting calculus becoming relatively inconsistent.

The concept of relative completeness applies only to calculi which contain a denial sign '$\sim$'. Another completeness concept may be defined which is applicable to all calculi, the concept of absolute completeness. We shall say that a calculus C is *absolutely complete* if C' is absolutely inconsistent where C' is like C except for containing as an extra axiom any well-formed formula of C which is not a theorem of C. It is easily seen that any calculus C which includes the formulae:

$$\ulcorner \psi \supset (\sim\psi \supset \varphi)\urcorner,$$

among its theorems and the rule of Detachment among its rules of deduction, is relatively complete if and only if absolutely complete.

Version I of the sentential calculus is both relatively and absolutely complete; version II of the same calculus, the quantificational calculus, and the identity calculus, however, are neither relatively nor absolutely complete.

The two concepts of relative and absolute completeness, being defined in terms of a syntactical concept, that of theoremhood, are syntactical concepts. A semantical concept of completeness may also be defined whereby a calculus C is complete if all the valid (or true, as the case may be) formulae of C are provable as theorems of C. We shall see below that version II of the sentential calculus, the quantificational calculus, and the identity calculus are semantically complete.

3. *Decidability.* We shall next say that a calculus C has a *decision procedure* or, more briefly, is *decidable*, if there exists a mechanical procedure for deciding whether any well-formed formula of C is provable or not as a theorem of C. The sentential calculus, for instance, is decidable; the truth-table method enables us to decide mechanically whether any well-formed formula of the sentential calculus is a tautology or not; but, as we shall establish below, a well-formed formula of the sentential calculus is a tautology if and only if it is provable as a theorem of the sentential calculus; the truth-table method thus enables us to decide mechanically whether any well-formed formula of the sentential calculus is provable or not as a theorem of the sentential calculus.

4. *Independence*:

(*a*) We shall say that an *axiom* φ of a calculus C is *independent* if φ is not provable as a theorem of C', where C' is like C except for containing as its axioms all the axioms of C but φ;

(*b*) We shall say that a *rule of deduction R* of a calculus C is *independent* if there exists at least one well-formed formula φ of C which is provable as a theorem of C but is not provable as a theorem of C', where C' is like C except for containing as its rules of deduction all the rules of deduction of C but R. The axioms and rules of deduction of our three calculi, the sentential calculus, the quantificational calculus, and the identity calculus, for instance, are independent.

*43. THE SENTENTIAL CALCULUS: PRELIMINARY METATHEOREMS

We shall prove in this section one preliminary metatheorem: "A well-formed formula φ is a theorem of the sentential calculus, version II, if and only if φ is a tautology." Half of it is readily proved as MT180; the other half (MT182) will be reached after one lemma (MT181).

MT180: If a well-formed formula φ is a theorem of the sentential calculus, version II, then φ is a tautology.

Proof: To establish MT180 we need only show that (*a*) the axioms of the sentential calculus, version II, are tautologies and that (*b*) any formula obtained from two tautologies by application of the rule of Detachment is a tautology. (*a*) is easily checked; as for (*b*) note that, by the truth-table for '$\supset$', a conditional $\ulcorner\varphi \supset \psi\urcorner$ whose antecedent φ is assigned 'T' throughout cannot itself be assigned 'T' throughout unless its consequent ψ is assigned 'T' throughout; hence if ψ is obtained from φ and $\ulcorner\varphi \supset \psi\urcorner$ by application of the rule of Detachment, and φ and $\ulcorner\varphi \supset \psi\urcorner$ are tautologies, then so is ψ.

Before turning to the next metatheorem let us consider the following truth-table:

p	q	$p \supset q$
T	T	T
F	T	T
T	F	F
F	F	T

This truth-table may be interpreted to say that:

(*a*) if 'p' and 'q' are true, then '$p \supset q$' is true;
(*b*) if 'p' is false and 'q' is true, then '$p \supset q$' is true;
(*c*) if 'p' is true and 'q' is false, then '$p \supset q$' is false;
(*d*) if 'p' and 'q' are false, then '$p \supset q$' is true;

or, in symbols:

$$
\begin{array}{ll}
(a) & (p \,.\, q) \supset (p \supset q); \\
(b) & (\sim p \,.\, q) \supset (p \supset q); \\
(c) & (p \,.\, \sim q) \supset \sim(p \supset q); \\
(d) & (\sim p \,.\, \sim q) \supset (p \supset q).
\end{array}
$$

Generalizing upon this result, let us consider a well-formed formula φ made up of n sentential dummies: $\zeta_1, \zeta_2, \ldots, \zeta_n$. Its truth-table will look as follows:

ζ_1	ζ_2	$\ldots$	ζ_n	φ
T	T		T	
F	T		T	
T	F		T	
F	F		T	
.	.		.	
.	.		.	
F	F		F	

and may be treated as a sequence of 2^n conditionals:

$$
\begin{array}{l}
\ulcorner(\zeta_1 \,.\, \zeta_2 \,.\, \ldots \,.\, \zeta_n) \supset \quad \urcorner, \\
\ulcorner(\sim\zeta_1 \,.\, \zeta_2 \,.\, \ldots \,.\, \zeta_n) \supset \quad \urcorner, \\
\ulcorner(\zeta_1 \,.\, \sim\zeta_2 \,.\, \ldots \,.\, \zeta_n) \supset \quad \urcorner, \\
\ulcorner(\sim\zeta_1 \,.\, \sim\zeta_2 \,.\, \ldots \,.\, \zeta_n) \supset \quad \urcorner, \\
\qquad\qquad . \\
\qquad\qquad . \\
\ulcorner(\sim\zeta_1 \,.\, \sim\zeta_2 \,.\, \ldots \,.\, \sim\zeta_n) \supset \quad \urcorner,
\end{array}
$$

whose blanks are filled by φ or by $\ulcorner\sim\varphi\urcorner$ depending upon whether φ is assigned the truth-value 'T' or the truth-value 'F' at the corresponding row of the table.

The purpose of the next metatheorem is to show that (a)–(d) and, more generally, the 2^n conditionals displayed in the preceding paragraph are theorems of the sentential calculus, version II.

MT181: Let $\zeta_1, \zeta_2, \ldots, \zeta_n$ be sentential dummies among which occur all the sentential dummies of φ; let $\psi_1, \psi_2, \ldots, \psi_n$ be well-formed formulae such that for m i's $(0 \leq m \leq n)$ ψ_i is ζ_i and for the remaining $n - m$ i's ψ_i is $\ulcorner\sim\zeta_1\urcorner$; let the truth-value 'T' be assigned to all ζ_i's such that ψ_i is ζ_i and the truth-value 'F' be assigned to all ζ_i's such that ψ_i is $\ulcorner\sim\psi_i\urcorner$; and let the truth-value of φ be computed by the truth-table method.

(*a*) If the truth-value of φ is 'T', then

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \varphi\urcorner \tag{1}$$

is a theorem of the sentential calculus, version II;

(*b*) If the truth-value of φ is 'F', then

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \sim\varphi\urcorner \tag{2}$$

is a theorem of the sentential calculus, version II.

Proof: We shall first show that (1) and (2) hold in case φ consists of 1 primitive sign; we shall next show that if (1) and (2) hold in case φ consists of m or fewer primitive signs, then they also hold in case φ consists of $m + 1$ primitive signs.

Case 1: φ consists of one primitive sign, a sentential dummy ζ_i. ψ_i is derivable from the assumption formulae $\psi_1, \psi_2, \ldots, \psi_n$; hence

$$\ulcorner\psi_1 \supset (\psi_2 \supset (\ldots \supset (\psi_n \supset \psi_i) \ldots))\urcorner$$

is a theorem by R5, and

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \psi_i\urcorner$$

is in turn a theorem by MT150 and $n - 1$ applications of R6. If, on the one hand, ψ_i is ζ_i, then ψ_i is φ, and hence

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \varphi\urcorner$$

is a theorem; but in this case ζ_i and hence φ are assigned the truth-value 'T'. If, on the other hand, ψ_i is $\ulcorner\sim\zeta_i\urcorner$, then ψ_i is $\ulcorner\sim\varphi\urcorner$, and hence

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \sim\varphi\urcorner$$

is a theorem; but in this case ζ_i and hence φ are assigned the truth-value 'F'. Q.E.D.

Case 2: φ consists of $m + 1$ signs; then φ is $\ulcorner\sim\chi\urcorner$ for some formula χ, or $\ulcorner\chi \vee \omega\urcorner$ for some formulae χ and ω, where χ and ω consist of m or fewer signs.

Subcase (2a): φ is $\ulcorner\sim\chi\urcorner$; then (1) becomes:

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \sim\chi\urcorner,$$

and (2) becomes:

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \sim\sim\chi\urcorner.$$

If, on the one hand, $\ulcorner\sim\chi\urcorner$ has the truth-value 'T', then χ has the truth-value 'F' and

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \sim\chi\urcorner$$

is a theorem. If, on the other hand, $\ulcorner \sim\chi \urcorner$ has the truth-value 'F', then χ has the truth-value 'T' and

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \chi\urcorner$$

is a theorem; but $\ulcorner\chi \supset \sim\sim\chi\urcorner$ is also a theorem (MT122); hence by MT111 and R1

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \sim\sim\chi\urcorner$$

is also a theorem. Q.E.D.

Subcase (2b): φ is $\ulcorner\chi \vee \omega\urcorner$; then (1) becomes:

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset (\chi \vee \omega)\urcorner,$$

and (2) becomes:

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \sim(\chi \vee \omega)\urcorner.$$

If, on the one hand, $\ulcorner\chi \vee \omega\urcorner$ has the truth-value 'T', then at least one of χ and ω has the truth-value 'T' and at least one of

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \chi\urcorner$$

and

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \omega\urcorner$$

is a theorem; but $\ulcorner\chi \supset (\chi \vee \omega)\urcorner$ and $\ulcorner\omega \supset (\chi \vee \omega)\urcorner$ are also theorems (MA102 and MT105); hence by MT111 and R1

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset (\chi \vee \omega)\urcorner$$

is also a theorem. If, on the other hand, $\ulcorner\chi \vee \omega\urcorner$ has the truth-value 'F', then both χ and ω have the truth-value 'F' and both

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \sim\chi\urcorner$$

and

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \sim\omega\urcorner$$

are theorems; but if

$$\ulcorner(\zeta_1 \,.\, \zeta_2 \,.\, \ldots \,.\, \psi_n) \supset \sim\chi\urcorner$$

and

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \sim\omega\urcorner$$

are theorems, then by R4, MT143, and R6,

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset (\sim\chi \,.\, \sim\omega)\urcorner$$

is a theorem; and if

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset (\sim\chi \,.\, \sim\omega)$$

is a theorem, then by MT162 and R6

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \sim(\chi \vee \omega)\urcorner$$

is a theorem. Q.E.D.

Before turning to MT182 let us consider the truth-table of a sample tautology, say, '$p \supset (q \supset p)$':

p	q	$p \supset (q \supset p)$	
T	T	T	T
F	T	T	F
T	F	T	T
F	F	T	T

We know from MT181 that the four conditionals:

$$(a)\ (p \,.\, q) \supset (p \supset (q \supset p)),$$
$$(b)\ (\sim p \,.\, q) \supset (p \supset (q \supset p)),$$
$$(c)\ (p \,.\, \sim q) \supset (p \supset (q \supset p)),$$

and

$$(d)\ (\sim p \,.\, \sim q) \supset (p \supset (q \supset p)),$$

are theorems. Now if (a) and (c) are theorems, so is

$$((p \,.\, q) \vee (p \,.\, \sim q)) \supset (p \supset (q \supset p));$$

hence so is

$$(p \,.\, (q \vee \sim q)) \supset (p \supset (q \supset p));$$

hence so is

$$(q \vee \sim q) \supset (p \supset (p \supset (q \supset p)));$$

hence so is

$$p \supset (p \supset (q \supset p)).$$

If, on the other hand, (b) and (d) are theorems, so is:

$$((\sim p \,.\, q) \vee (\sim p \,.\, \sim q)) \supset (p \supset (q \supset p));$$

hence so is

$$(\sim p \,.\, (q \vee \sim q)) \supset (p \supset (q \supset p));$$

hence so is

$$(q \vee \sim q) \supset (\sim p \supset (p \supset (q \supset p)));$$

hence so is

$$\sim p \supset (p \supset (q \supset p)).$$

But if

$$p \supset (p \supset (q \supset p))$$

and

$$\sim p \supset (p \supset (q \supset p))$$

are theorems, then so is

$$(p \vee \sim p) \supset (p \supset (q \supset p)),$$

and hence so is tautology

$$p \supset (q \supset p).$$

MT182 extends this result to any tautology φ:

MT182: If φ is a tautology, then φ is a theorem of the sentential calculus, version II.[4]

Proof: Let $\zeta_1, \zeta_2, \ldots, \zeta_n$ and $\psi_1, \psi_2, \ldots, \psi_n$ be as in metatheorem MT181. Since φ is a tautology, then by MT181

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \varphi\urcorner \tag{a}$$

is a theorem for every one of the 2^n sets of ψ_i's such that ψ_i is either ζ_i or $\ulcorner\sim\zeta_i\urcorner$.

(1): $\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \zeta_n) \supset \varphi\urcorner$ (a, where ψ_n is ζ_n)
(2): $\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \sim\zeta_n) \supset \varphi\urcorner$ (a, where ψ_n is $\ulcorner\sim\zeta_n\urcorner$)
(3): $\ulcorner((\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \zeta_n) \vee (\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \sim\zeta_n)) \supset \varphi\urcorner$
(I, (A, 1, 2), 145)
(4): $\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, (\zeta_n \vee \sim\zeta_n)) \supset \varphi\urcorner$ (I, 3, 141)
(5): $\ulcorner(\zeta_n \vee \sim\zeta_n) \supset ((\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_{n-1}) \supset \varphi)\urcorner$
(I, 4, 151)
(6): $\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_{n-1}) \supset \varphi\urcorner$ (108, 5)
Similarly:
(7): $\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \zeta_{n-1}) \supset \varphi\urcorner$ (6, where ψ_{n-1} is ζ_{n-1})
(8): $\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \sim\zeta_{n-1}) \supset \varphi\urcorner$ (6, where ψ_{n-1} is $\ulcorner\sim\zeta_{n-1}\urcorner$)
.
.
(12): $\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_{n-2}) \supset \varphi\urcorner$
.
.
$(6n-6)$: $\ulcorner\psi_1 \supset \varphi\urcorner$[5]
$(6n-5)$: $\ulcorner\zeta_1 \supset \varphi\urcorner$ ($6n-6$, where ψ_1 is ζ_1)
$(6n-4)$: $\ulcorner\sim\zeta_1 \supset \varphi\urcorner$ ($6n-6$, where ψ_1 is $\ulcorner\sim\zeta_1\urcorner$)
$(6n-3)$: $\ulcorner(\zeta_1 \vee \sim\zeta_1) \supset \varphi\urcorner$ (I, (A, $6n-5$, $6n-4$),145)
$(6n-2)$: φ (108, $6n-3$)

[4]There exist several proofs of MT182; the proof given here, along with the proof of MT181, is essentially due to B. Rosser.

[5]Note that $6n-5$ in '$(6n-5)$' is 6 times n minus 5, where n is the number of sentential dummies ζ_i.

MT180 and MT182 together yield:

MT183: A well-formed formula φ is a theorem of the sentential calculus, version II, if and only if φ is a tautology.

MT183 is a syntactical metatheorem; a semantical analogue of it is next proved as MT184:

MT184: A well-formed formula φ is a theorem of the sentential calculus, version II, if and only if φ is sententially valid.

Proof: If φ is assigned the truth-value 'T' whatever truth-values be assigned to its sentential dummies $\zeta_1, \zeta_2, \ldots, \zeta_n$, then φ yields truths whether true or false statements be substituted for $\zeta_1, \zeta_2, \ldots, \zeta_n$, and *vice-versa*. But φ is a theorem of the sentential calculus, version II, if and only if φ is a tautology. Hence φ is a theorem of the sentential calculus, version II, if and only if φ is sententially valid.

Analogues of MT180–MT184 hold for version I of the sentential calculus; their proof is left to the reader.

*44. THE SENTENTIAL CALCULUS: CONSISTENCY, COMPLETENESS, AND DECIDABILITY

We distinguished above between two types of consistency: relative consistency and absolute consistency. A calculus C was said to be *relatively consistent* if of any two well-formed formulae φ and $\ulcorner\sim\varphi\urcorner$ of C at most one is a theorem of C.

MT185: Version II of the sentential calculus is relatively consistent.

Proof: Of any two well-formed formulae φ and $\ulcorner\sim\varphi\urcorner$ of version II at most one is a tautology; hence, by MT183, of any two well-formed formulae φ and $\ulcorner\sim\varphi\urcorner$ of version II at most one is a theorem of version II.

A calculus C was next said to be *absolutely consistent* if at least one well-formed formula of C is not a theorem of C.

MT186: Version II of the sentential calculus is absolutely consistent.

Proof: The well-formed formula 'p' is not a tautology and, hence, by MT180, not a theorem of version II.

Analogues of MT185 and MT186 hold for version I of the sentential calculus; their proof is left to the reader.

Let us note that a given calculus may be inconsistent either on account of its axioms or on account of its rules of deduction. Versions I and II of the sentential calculus, for instance, would be inconsistent if they included among their axioms any such formula as '$\sim(p \supset p)$', or included among their rules of deduction any such rule as "From φ and $\ulcorner\varphi \supset \psi\urcorner$ one may deduce $\ulcorner\sim\psi\urcorner$."

We distinguished above between two types of completeness: relative completeness and absolute completeness. A calculus C was said to be *relatively complete* if C' is relatively inconsistent, where C' is like C except for containing as an extra axiom any well-formed formula of C which is not a theorem of C.

Note: In the next two proofs, '*MT181' and '*MT183' respectively refer to the version I analogues of MT181 and MT183.

MT187: Version I of the sentential calculus is relatively complete.

Proof: If a well-formed formula φ is not a theorem of version I, then by *MT183 φ is not a tautology; but if φ is not a tautology, then by *MT181 there is at least one set of well-formed formulae $\psi_1, \psi_2, \ldots, \psi_n$ such that

$$\ulcorner(\psi_1 \,.\, \psi_2 \,.\, \ldots \,.\, \psi_n) \supset \sim\varphi\urcorner$$

or, equivalently,

$$\ulcorner\psi_1 \supset (\psi_2 \supset (\ldots \supset (\psi_n \supset \sim\varphi) \ldots))\urcorner \tag{1}$$

is a theorem of version I. Let

$$\ulcorner\psi'_1 \supset (\psi'_2 \supset (\ldots \supset (\psi'_n \supset \sim\varphi') \ldots))\urcorner \tag{2}$$

be the result of replacing each ζ_i in (1) by '$p \supset p$' or '$\sim(p \supset p)$' depending upon whether ψ_i is ζ_i or $\ulcorner\sim\zeta_i\urcorner$. '$p \supset p$' and '$\sim\sim(p \supset p)$', being tautologies, are by *MT183 theorems of version I. Hence by (2) and n applications of the rule of Detachment $\ulcorner\sim\varphi'\urcorner$ is a theorem of version I. If, on the other hand, φ is appointed an extra axiom of version I, then by R3a φ' is a theorem of version I. If φ is appointed as an extra axiom of version I, both φ' and $\ulcorner\sim\varphi'\urcorner$ are therefore provable as theorems, and the resulting calculus is relatively inconsistent.

A calculus C was next said to be *absolutely complete* if C' is absolutely inconsistent where C' is like C except for containing as an extra axiom any well-formed formula of C which is not a theorem of C.

MT188: Version I of the sentential calculus is absolutely complete.

Proof: If any well-formed formula φ which is not a theorem of version I is appointed as an extra axiom of version I, then by MT187 both φ', and $\ulcorner\sim\varphi'\urcorner$ are provable as theorems of version I. But $\ulcorner\varphi' \supset (\sim\varphi' \supset \psi)\urcorner$, where ψ is any well-formed formula, is a tautology and, hence, by *MT183 a theorem of version I. If any well-formed formula φ which is not a theorem of version I is appointed as an extra axiom of version I, then ψ is provable as a theorem, and the resulting calculus is absolutely inconsistent.

Let us note that a given calculus may be incomplete either on account of its axioms or on account of its rules of deduction. Version I of the sentential calculus, for instance, would be incomplete if it did not include any one of A100–A103 among its axioms or either one of R1 and R3a among its rules of deduction.

MT187 fails for version II of the sentential calculus; note indeed that if version II is enlarged to contain any individual formula φ as an extra axiom, then R3a is no longer provable as a derived rule of deduction and φ' need no longer be provable as a theorem (cf. page 123n). Since MT187 fails for version II, MT188 also does. Version II is, however, semantically complete; it follows indeed from MT184 that if a well-formed formula φ of version II is sententially valid, then φ is a theorem of version II.

A calculus C was said to be *decidable* if there exists a mechanical procedure for deciding whether any well-formed formula of C is a theorem of C or not.

MT189: Version II of the sentential calculus is decidable.

Proof: There exists a mechanical procedure for deciding whether any well-formed formula of version II is a tautology or not: the truth-table method. Hence by MT183 there exists a mechanical procedure for deciding whether any well-formed formula of version II is a theorem of version II or not.

The analogue of MT189 for version I of the sentential calculus also holds, as the reader may check.

*45. THE SENTENTIAL CALCULUS: INDEPENDENCE

An *axiom* φ of a calculus C was said to be *independent* if φ is not a theorem of C', where C' is like C except for containing as its axioms all the axioms of C but φ.

MT190: Axioms MA100–MA103 of version II of the sentential calculus are independent.

Proof:

Step 1: Let '$\sim$' and 'v' obey the following truth-tables:

φ	$\ulcorner \sim\varphi \urcorner$
0	1
1	0
2	2

φ	ψ	$\ulcorner \varphi \vee \psi \urcorner$
0	0	0
1	0	0
2	0	0
0	1	0
1	1	1
2	1	2
0	2	0
1	2	2
2	2	0

and let a well-formed formula φ be said to be a *0-formula* if it has the truth-value '0' for any assignment of the three truth-values '0', '1', and '2' to its sentential dummies. It is easily checked that MA101–MA103 are 0-formulae and that any formula obtainable by R1 from two 0-formulae is a 0-formula. But '$\sim(p \vee p) \vee p$' has the truth-value '2' when 'p' is assigned the truth-value '2'; '$\sim(p \vee p) \vee p$' is therefore not provable from MA101–MA103 by R1.

Step 2: Let '$\sim$' and 'v' obey the following truth-tables:

φ	$\ulcorner \sim\varphi \urcorner$
0	1
1	0
2	3
3	2

φ	ψ	$\ulcorner \varphi \vee \psi \urcorner$
0	0	0
1	0	0
2	0	0
3	0	0
0	1	0
1	1	1
2	1	1
3	1	1
0	2	0
1	2	1
2	2	2
3	2	2
0	3	0
1	3	1
2	3	2
3	3	3

and let a well-formed formula φ be said to be a *0-formula* if it has the truth-value '0' for any assignment of the four truth-values '0', '1', '2,' and '3' to its sentential dummies. It is easily checked that MA100, MA102, and MA103 are 0-formulae and that any formula obtainable by R1 from two 0-formulae is a 0-formula. But '$\sim p \vee (p \vee q)$' has the truth-value '1' when 'p' is assigned the truth-value '2' and 'q' is assigned the truth-value '1'; '$\sim p \vee (p \vee q)$' is therefore not provable from MA100, MA102, or MA103 by R1.

Step 3: Let '$\sim$' and '$\vee$' obey the following truth-tables:

φ	$\ulcorner \sim\varphi \urcorner$
0	1
1	0
2	0
3	2

φ	ψ	$\ulcorner \varphi \vee \psi \urcorner$
0	0	0
1	0	0
2	0	0
3	0	0
0	1	0
1	1	1
2	1	2
3	1	3
0	2	0
1	2	2
2	2	2
3	2	3
0	3	0
1	3	3
2	3	0
3	3	3

and let a well-formed formula φ be said to be a *0-formula* if it has the truth-value '0' for any assignment of the four truth-values '0', '1', '2', and '3' to its sentential dummies. It is easily checked that MA100, MA101, and MA103 are 0-formulae and that any formula obtainable by R1 from two 0-formulae is a 0-formula. But '$\sim(p \vee q) \vee (q \vee p)$' has the truth-value '3' when 'p' is assigned the truth-value '2' and 'q' is assigned the truth-value '3'; '$\sim(p \vee q) \vee (q \vee p)$' is therefore not provable from MA101, MA101, or MA103 by R1.

Step 4: Let '$\sim$' and 'v' obey the following truth-tables:

φ	$\ulcorner\sim\varphi\urcorner$
0	1
1	0
2	0
3	2

φ	ψ	$\ulcorner\varphi \vee \psi\urcorner$
0	0	0
1	0	0
2	0	0
3	0	0
0	1	0
1	1	1
2	1	2
3	1	3
0	2	0
1	2	2
2	2	2
3	2	0
0	3	0
1	3	3
2	3	0
3	3	3

and let a well-formed formula φ be said to be a *0-formula* if it has the truth-value '0' for any assignment of the four truth-values '0', '1', '2', and '3' to its sentential dummies. It is easily checked that MA100–MA102 are 0-formulae and that any formula obtainable by R1 from two 0-formulae is a 0-formula. But '$\sim(\sim p \vee q) \vee (\sim(r \vee p) \vee (r \vee q))$' has the truth-value '2' when 'p' is assigned the truth-value '3', 'q' the truth-value '1', and 'r' the truth-value '2'; '$\sim(\sim p \vee q) \vee (\sim(r \vee p) \vee (r \vee q))$' is therefore not provable from MA100–MA102 by R1.

The analogue of MT190 for version I of the sentential calculus is easily seen to hold.

A *rule of deduction R* of a calculus C was next said to be *independent* if there exists at least one theorem of C which is not a theorem of C', where C' is like C except for containing as its rules of deduction all the rules of deduction of C but R.

MT191: Rule of deduction R1 of version II of the sentential calculus is independent.

Proof: Let version II$'$ contain MA100–MA103 as its axioms but no rule of deduction; then no formula shorter than any of MA100–MA103 will be provable as a theorem of version II$'$. But theorem '$(p \vee \sim p)$' of version II is shorter than any of MA100–MA103; theorem '$(p \vee \sim p)$' of version II is therefore not provable as a theorem of version II$'$.

It can be shown likewise that the rules of deduction of version I are independent. Let version I′ contain A100–A103 as its axioms and R3a as its rule of deduction; then no formula shorter than any of A001–A103 will be provable as a theorem of version I′. But theorem '$(p \vee \sim p)$' of version I is shorter than any of A100–A103; theorem '$(p \vee \sim p)$' of version I is therefore not provable as a theorem of version I′. Let, on the other hand, version I″ contain A100–A103 as its axioms and R1 as its rule of deduction; then no formula longer than any of A100–A103 will be provable as a theorem of version I″. But theorem '$\sim(\sim p \vee (q \vee s)) \vee (\sim(r \vee p) \vee (r \vee (q \vee s)))$' of version I is longer than any of A100–A103; theorem '$\sim(\sim p \vee (q \vee s)) \vee (\sim(r \vee p) \vee (r \vee (q \vee s)))$' of version I is therefore not provable as a theorem of version I″.[6]

*46. THE QUANTIFICATIONAL CALCULUS: CONSISTENCY

The quantificational calculus is easily shown to be both relatively and absolutely consistent.

Let φ^+ be the result of deleting all the quantifiers and all the argument sequences $\ulcorner(\alpha_1, \alpha_2, \ldots, \alpha_n)\urcorner$ occurring in a formula φ. Let φ^* be the result of replacing each occurrence of κ_i in φ^+, where κ_i is the alphabetically i-th predicate dummy to occur in φ^+, by an occurrence of the alphabetically i-th sentential dummy foreign to φ^+. We shall call φ^* *the sentential associate of* φ.

Examples: Let φ be '$(x)F(x) \supset F(y)$'; φ^+ will be '$F \supset F$', and φ^* will be '$p \supset p$'. Let φ be '$(x)(p \supset F(x)) \supset (p \supset (x)F(x))$'; φ^+ will be '$(p \supset F) \supset (p \supset F)$', and φ^* will be '$(p \supset q) \supset (p \supset q)$'. Let φ be '$p \supset (p \vee q)$'; φ^+ and φ^* will be '$p \supset (p \vee q)$'.

MT250: If φ is a theorem of the quantificational calculus, then φ^* is a tautology.

Proof:

1. The sentential associates of MA100–MA103 and MA200–MA201 are formulae of the form:

$$\ulcorner(\varphi^* \vee \varphi^*) \supset \varphi^*\urcorner,$$
$$\ulcorner\varphi^* \supset (\varphi^* \vee \psi^*)\urcorner,$$
$$\ulcorner(\varphi^* \vee \psi^*) \supset (\psi^* \vee \varphi^*)\urcorner,$$
$$\ulcorner(\varphi^* \supset \psi^*) \supset ((\chi^* \vee \varphi^*) \supset (\chi^* \vee \psi^*))\urcorner,$$
$$\ulcorner\varphi^* \supset \varphi^*\urcorner,$$

and

$$\ulcorner(\varphi^* \supset \psi^*) \supset (\varphi^* \supset \psi^*)\urcorner,$$

all of which are tautologies.

[6] As shown in chapter three, A100–A103 may, however, be replaced by metaaxioms, in which case R3a may be dispensed with.

2. R1 and R2 never lead from formulae whose sentential associates are tautologies to formulae whose sentential associates are not tautologies. This is immediate in the case of R2; as for R1, note that φ^* and $\ulcorner\varphi^* \supset \psi^*\urcorner$ cannot be tautologies without φ^* being also a tautology.

3. It follows from 1, 2, and the definition of a theorem that if φ is a theorem of the quantificational calculus, then its sentential associate φ^* is a tautology.

The following two metatheorems are corollaries of MT250:

MT251: The quantificational calculus is relatively consistent.

Proof: If, on one hand, φ is a theorem of the quantificational calculus, then by MT250 φ^* is a tautology; but if φ^* is a tautology, then $\ulcorner\sim\varphi^*\urcorner$ is not a tautology and, by MT250, $\ulcorner\sim\varphi\urcorner$ is not a theorem of the quantificational calculus. If, on the other hand, $\ulcorner\sim\varphi\urcorner$ is a theorem of the quantificational calculus, then by MT250 $\ulcorner\sim\varphi^*\urcorner$ is a tautology; but if $\ulcorner\sim\varphi^*\urcorner$ is a tautology, then φ^* is not a tautology and, by MT250, φ is not a theorem of the quantificational calculus. Of any two formulae φ and $\ulcorner\sim\varphi\urcorner$ at most one will therefore be a theorem of the quantificational calculus.

MT252: The quantificational calculus is absolutely consistent.

Proof: The sentential associate '$\sim(p \supset p)$' of '$\sim((x)F(x) \supset (x)F(x))$' is not a tautology and, hence, by MT250 '$\sim((x)F(x) \supset (x)F(x))$' is not a theorem of the quantificational calculus.

*47. THE QUANTIFICATIONAL CALCULUS: COMPLETENESS

The quantificational calculus, though both relatively and absolutely consistent, is neither relatively nor absolutely complete. We proved in section 44 that version I of the sentential calculus becomes inconsistent if any sententially indeterminate or sententially contravalid formula is appointed as an extra axiom of the calculus. The quantificational calculus becomes likewise inconsistent if any quantificationally contravalid formula is appointed as an extra axiom of the calculus, but it may remain consistent after certain quantificationally indeterminate formulae are appointed as extra axioms of the calculus. The well-formed formula '$(\mathrm{E}x)F(x) \supset (x)F(x)$', for instance, is quantificationally indeterminate; yet it may be appointed as an extra axiom without the quantificational calculus becoming inconsistent. Note indeed that '$p \supset p$', the sentential associate of '$(\mathrm{E}x)F(x) \supset (x)F(x)$', is a tautology; hence if '$(\mathrm{E}x)F(x) \supset (x)F(x)$' is appointed as an extra axiom, the theorems of the quantificational calculus will again have tautologies as their sentential associates

and '$\sim((\mathrm{E}x)F(x) \supset (x)F(x))$', whose sentential associate is not a tautology, will not be provable as a theorem.

A semantical metatheorem, parallel to MT184, may, however, be proved about the quantificational calculus, namely: "A well-formed formula φ is a theorem of the quantificational calculus if and only if φ is quantificationally valid." Half of it: "If a well-formed formula φ is a theorem of the quantificational calculus, then φ is quantificationally valid," is obvious enough; it will be taken here for granted. The other half, however, from which follows as a corollary the semantical completeness of the quantificational calculus, is far from obvious; it was first proved in 1930 by the Austrian logician Kurt Gödel. We shall reproduce here the essentials of Gödel's proof; the missing details may be found in more advanced treatises.

The initial step of the proof consists in putting all the well-formed formulae of the quantificational calculus in standard form.

(*a*) A well-formed formula φ is said to be in *prenex normal form* when it is of the form:

$$\ulcorner(\mathrm{Q}\alpha_1)(\mathrm{Q}\alpha_2) \ldots (\mathrm{Q}\alpha_n)\psi\urcorner,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ ($n \geq 0$) are distinct argument variables, each one of $\ulcorner(\mathrm{Q}\alpha_i)\urcorner$ is either $\ulcorner(\alpha_i)\urcorner$ or $\ulcorner(\mathrm{E}\alpha_i)\urcorner$, and ψ is a quantifier free formula in which each one of the variables $\alpha_1, \alpha_2, \ldots, \alpha_n$ occurs at least once.

Examples: '$(x)(F(x) \supset F(x))$' and '$(y)(\mathrm{E}x)(F(x) \supset F(y))$' are in prenex normal form; '$(x)(y)F(x)$' and '$(y)((x)F(x) \supset F(y))$', on the other hand, are not.

It is easily shown that:

Lemma I: There exists a mechanical procedure whereby, given any well-formed formula φ, a well-formed formula φ' in prenex normal form can be found such that

$$\ulcorner\varphi \equiv \varphi'\urcorner$$

is a theorem;

we shall, however, leave lemma I without proof.

(*b*) A well-formed formula φ is said to be in *Skolem normal form* when it is of the form:

$$\ulcorner(\mathrm{E}\alpha_1)(\mathrm{E}\alpha_2) \ldots (\mathrm{E}\alpha_k)(\beta_1)(\beta_2) \ldots (\beta_m)\psi\urcorner,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ ($k \geq 0$), $\beta_1, \beta_2, \ldots, \beta_m$ ($m \geq 0$) are all the distinct argument variables occurring in ψ, and ψ is a quantifier free formula. A formula in Skolem normal form is thus a formula in prenex normal form all the argument variables of which (if any) are bound and all the existential quantifiers of which (if any) are initially placed.

Examples: '$(x)(F(x) \supset F(x))$' and '$(\mathrm{E}x)(y)(F(x) \supset F(y))$' are in Skolem normal form; '$(y)(\mathrm{E}x)(F(x) \supset F(y))$' and '$(\mathrm{E}x)(y)(F(x,y) \supset F(x,z))$', on the other hand, are not.
It can be shown that:

Lemma II: There exists a mechanical procedure whereby, given any well-formed formula φ in prenex normal form, a well-formed formula φ' in Skolem normal form can be found such that φ is a theorem if and only if φ' is a theorem.[7]

We shall not prove lemma II here; the reader may check, however, that '$(\mathrm{E}x)(\mathrm{E}y)(F(y) \supset F(x))$' and '$(\mathrm{E}x)(\mathrm{E}y)(z)((F(x) \supset G(x)) \supset (F(z) \supset G(y)))$' are both in Skolem normal form and that:

1. '$(x)F(x) \supset (\mathrm{E}x)F(x)$' is a theorem if and only if '$(\mathrm{E}x)(\mathrm{E}y)(F(y) \supset F(x))$' is;
2. '$(x)(F(x) \supset G(x)) \supset ((\mathrm{E}x)F(x) \supset (\mathrm{E}x)G(x))$' is a theorem if and only if '$(\mathrm{E}x)(\mathrm{E}y)(z)((F(x) \supset G(x)) \supset (F(z) \supset G(y)))$' is.

We shall use '$(\mathrm{E}x)(\mathrm{E}y)(F(y) \supset F(x))$' and '$(\mathrm{E}x)(\mathrm{E}y)(z)((F(x) \supset G(x)) \supset (F(z) \supset G(y)))$' below to illustrate the various steps of Gödel's argument.[8]

In view of lemma II we need only prove the weaker metatheorem:

MT253: If a well-formed formula φ is Skolem normal form is quantificationally valid, then φ is a theorem of the quantificational calculus;

the stronger metatheorem:

MT254: If a well-formed formula φ is quantificationally valid, then φ is a theorem of the quantificational calculus,

will follow as a corollary of MT253.

Gödel's proof of MT253 proceeds as follows. Corresponding to every well-formed formula:

$$\ulcorner(\mathrm{E}\alpha_1)(\mathrm{E}\alpha_2)\ldots(\mathrm{E}\alpha_k)(\beta_1)(\beta_2)\ldots(\beta_m)\varphi\urcorner \qquad (1),$$

in Skolem normal form, a sequence of formulae χ_n ($n = 1, 2, 3, \ldots$) is first constructed. It is next shown that:

(*a*) If, on one hand, χ_n is a tautology for some n, then (1) is a theorem of the quantificational calculus;

(*b*) If, on the other hand, χ_n is not a tautology for any n, then (1) is not quantificationally valid.

[7]Note the distinction between lemma I and lemma II. We can prove, in lemma I, that $\ulcorner\varphi \equiv \varphi'\urcorner$ is a theorem and hence that φ is a theorem if and only if φ' is; we can only prove, in lemma II, that φ is a theorem if and only if φ' is.

[8]The reader will find a proof of lemma I and lemma II in Hilbert and Ackermann's *Principles of Mathematical Logic*, pp. 83–87.

It follows from (*a*) and (*b*) that (1) is a theorem of the quantificational calculus or else is not quantificationally valid, and, hence, that if (1) is quantificationally valid, then it is a theorem of the quantificational calculus.

The formulae χ_n are constructed as follows:

(*a*) We first replace the argument variables 'w', 'x', 'y', 'z', and their accented variants by the following variables:

$$'x_0', \ 'x_1', \ 'x_2', \ 'x_3', \ \ldots, \ 'x_n', \ \ldots,$$

where 'n' is called *the index of* 'x_n'.

(*b*) We next form k-tuples out of the variables 'x_0', 'x_1', 'x_2', and so on, and order them: first, according to increasing index sums; second, lexicographically within each set of k-tuples having the same index sum.[9]

Examples: If $k = 1$, then we obtain the following sequence of k-tuples: (x_0), (x_1), (x_2), and so on. If $k = 2$, then we obtain the following sequence of k-tuples: (x_0,x_0), (x_0,x_1), (x_1,x_0), (x_0,x_2), (x_1,x_1), (x_2,x_0), and so on. If $k = 3$, then we obtain the following sequences of k-tuples: (x_0,x_0,x_0), (x_0,x_0,x_1), (x_0,x_1,x_0), (x_1,x_0,x_0), (x_0,x_0,x_2), (x_0,x_1,x_1), (x_0,x_2,x_0), (x_1,x_0,x_1), (x_1,x_1,x_0), (x_2,x_0,x_0), and so on.

(*c*) Let the n-th one of these k-tuples be:

$$(x_{n_1}, x_{n_2}, \ldots, x_{n_k}),$$

and let ψ_n, for each n from 1 on, be:

$$\ulcorner \varphi \frac{x_{n_1}, \ x_{n_2}, \ \ldots, \ x_{n_k}, \ x_{(n-1)m+1}, \ x_{(n-1)m+2}, \ \ldots, \ x_{(n-1)m+m}}{\alpha_1, \ \alpha_2, \ \ldots, \ \alpha_k, \quad \beta_1 \quad, \quad \beta_2 \quad, \ \ldots, \quad \beta_m} \urcorner^{10}.$$

Examples: Let (1) be '$(\mathrm{E}x)(\mathrm{E}y)(F(y) \supset F(x))$'; k here equals 2 and m equals 0. We get:

ψ_1: '$F(x_0) \supset F(x_0)$',
ψ_2: '$F(x_0) \supset F(x_1)$',
ψ_3: '$F(x_1) \supset F(x_0)$',
ψ_4: '$F(x_0) \supset F(x_2)$', and so on.

Let (1) be '$(\mathrm{E}x)(\mathrm{E}y)(z)((F(x) \supset G(x)) \supset (F(z) \supset G(y)))$'; k here equals 2 and m equals 1. We get:

ψ_1: '$(F(x_0) \supset G(x_0)) \supset (F(x_1) \supset G(x_0))$',
ψ_2: '$(F(x_0) \supset G(x_0)) \supset (F(x_2) \supset G(x_1))$',
ψ_3: '$(F(x_1) \supset G(x_1)) \supset (F(x_3) \supset G(x_0))$',
ψ_4: '$(F(x_0) \supset G(x_0)) \supset (F(x_4) \supset G(x_2))$', and so on.

[9] k, let us recall, is the number of existentially quantified variables of (1).
[10] '$x_{(n-1)m+m}$' will from now on be condensed to read 'x_{nm}'.

(*d*) Let χ_n be $\ulcorner\zeta_1 \vee \psi_2 \vee \ldots \vee \psi_n\urcorner$.

Example: If (1) is '$(\mathrm{E}x)(\mathrm{E}y)(F(y) \supset F(x))$', then

χ_1 will be '$F(x_0) \supset F(x_0)$',
χ_2 will be '$(F(x_0) \supset F(x_0)) \vee (F(x_0) \supset F(x_1))$',
χ_3 will be '$(F(x_0) \supset F(x_0)) \vee (F(x_0) \supset F(x_1)) \vee (F(x_1) \supset F(x_0))$', and so on.[11]

We meet at this point the following alternative:

1. there is one n such that χ_n is a tautology,
or 2. there is no n such that χ_n is a tautology.

We shall now prove that in the first case (1) is a theorem of the quantificational calculus and that in the second (1) is not quantificationally valid.

Case 1: If χ_n is a tautology, then by MT182 χ_n is a theorem of the sentential calculus and hence a theorem of the quantificational calculus; but if χ_n is a theorem of the quantificational calculus, then the result: $\ulcorner(x_0)(x_1) \ldots (x_{nm})\chi_n\urcorner$, of universally quantifying all the free variables of χ_n is a theorem of the quantificational calculus. Let $\ulcorner(x_0)(x_1) \ldots (x_{nm})\chi_n\urcorner$ be abbreviated $\ulcorner\mathrm{U}\chi_n\urcorner$. We shall now show that if $\ulcorner\mathrm{U}\chi_n\urcorner$ is a theorem of the quantificational calculus, then so is (1). The proof of $\ulcorner\mathrm{U}\chi_n \supset (1)\urcorner$ will proceed by mathematical induction; we shall first show that

$$\ulcorner\mathrm{U}\chi_1 \supset (1)\urcorner$$

is a theorem; we shall next show that if

$$\ulcorner\mathrm{U}\chi_{n-1} \supset (1)\urcorner$$

is a theorem for any natural number n, then so is

$$\ulcorner\mathrm{U}\chi_n \supset (1)\urcorner.$$

Subcase (1a): $\ulcorner\mathrm{U}\chi_1 \supset (1)\urcorner$ is a theorem.

Proof: By repeated applications of MT217

$$\ulcorner(\beta_1) \ldots (\beta_m)\varphi\frac{z_0, \ldots, z_0}{\alpha_1, \ldots, \alpha_k} \supset (\mathrm{E}x_1) \ldots (\mathrm{E}x_k)(\beta_1) \ldots (\beta_m)\varphi\frac{x_1, \ldots, x_k}{\alpha_1, \ldots, \alpha_k}\urcorner \tag{2}$$

is a theorem. But $\ulcorner\mathrm{U}\chi_1\urcorner$ is of the form:

$$\ulcorner(x_0) \ldots (x_m)\varphi\frac{x_0, \ldots, x_0, x_1, \ldots, x_m}{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m}\urcorner;$$

[11]Observe that none of the argument variables '$x_{(n-1)m+1}$', '$x_{(n-1)m+2}$', ..., 'x_{nm}' occurs in any ψ_j ($j < n$), but that all the argument variables 'x_{n_1}', 'x_{n_2}', ..., 'x_{n_k}' occur in some ψ_j ($j < n$).

hence, by MA200,

$$\ulcorner \mathrm{U}\chi_1 \supset (x_1) \ldots (x_m)\varphi\frac{z_0, \ldots, z_0, x_1, \ldots, x_m}{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m}\urcorner \tag{3}$$

is a theorem. But if (3) is a theorem, then by R7

$$\ulcorner \mathrm{U}\chi_1 \supset (\beta_1) \ldots (\beta_m)\varphi\frac{z_0, \ldots, z_0}{\alpha_1, \ldots, \alpha_k}\urcorner \tag{4}$$

is a theorem. By (2), (4), and MT111,

$$\ulcorner \mathrm{U}\chi_1 \supset (\mathrm{E}x_1) \ldots (\mathrm{E}x_k)(\beta_1) \ldots (\beta_m)\varphi\frac{x_1, \ldots, x_k}{\alpha_1, \ldots, \alpha_k}\urcorner$$

is therefore a theorem and, hence, by R7

$$\ulcorner \mathrm{U}\chi_1 \supset (\mathrm{E}\alpha_1) \ldots (\mathrm{E}\alpha_k)(\beta_1) \ldots (\beta_m)\varphi\urcorner,$$

that is,

$$\ulcorner \mathrm{U}\chi_1 \supset (1)\urcorner,$$

is a theorem. Q.E.D.

Subcase (1b): If $\ulcorner \mathrm{U}\chi_{n-1} \supset (1)\urcorner$ is a theorem, then so is $\ulcorner \mathrm{U}\chi_n \supset (1)\urcorner$.

Proof: $\ulcorner \mathrm{U}\chi_n\urcorner$ is of the form: $\ulcorner (x_0) \ldots (x_{nm})\chi_n\urcorner$ or $\ulcorner (x_0) \ldots (x_{nm})(\chi_{n-1} \vee \psi_n)\urcorner$, where ψ_n is:

$$\ulcorner \varphi\frac{x_{n_1}, \ldots, x_{n_k}, x_{(n-1)m+1}, \ldots, x_{nm}}{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m}\urcorner.$$

Since, as noted above, the argument variables '$x_{(n-1)m+1}$', . . . , 'x_{nm}' do not occur in χ_{n-1}, we obtain by repeated applications of MT230:

$$\ulcorner (x_{(n-1)m+1}) \ldots (x_{nm})(\chi_{n-1} \vee \psi_n) \equiv (\chi_{n-1} \vee (x_{(n-1)m+1}) \ldots (x_{nm})\psi_n)\urcorner,$$

and hence obtain by MT156:

$$\ulcorner (x_{(n-1)m+1}) \ldots (x_{nm})(\chi_{n-1} \vee \psi_n) \supset (\chi_{n-1} \vee (x_{(n-1)m+1}) \ldots (x_{nm})\psi_n\urcorner \tag{5}$$

But if (5) is a theorem, then by repeated applications of R2 and MT204

$$\ulcorner \mathrm{U}\chi_n \supset (x_0) \ldots (x_{(n-1)m})(\chi_{n-1} \vee (x_{(n-1)m+1}) \ldots (x_{nm})\psi_n)\urcorner$$

is a theorem and, hence, by R7,

$$\ulcorner \mathrm{U}\chi_n \supset (x_0) \ldots (x_{(n-1)m})(\chi_{n-1} \vee (\beta_1) \ldots (\beta_m)\frac{x_{n_1}, \ldots, x_{n_k}}{\alpha_1, \ldots, \alpha_k}\urcorner \tag{6}$$

is a theorem.

On the other hand, by repeated applications of MT217,

$$\ulcorner (\beta_1) \ldots (\beta_m)\varphi\frac{x_{n_1}, \ldots, x_{n_k}}{\alpha_1, \ldots, \alpha_k} \supset (1)\urcorner$$

is a theorem and, hence, by MA103,

$$\ulcorner\left(\mathrm{U}\chi_{n-1} \vee (\beta_1) \ldots (\beta_m)\varphi\frac{x_{n_1}, \ldots, x_{nk}}{\alpha_1, \ldots, \alpha_k}\right) \supset (\mathrm{U}\chi_{n-1} \vee (1))\urcorner \tag{7}$$

is a theorem. But if (7) is a theorem, then by repeated applications of R2 and MT204

$$\ulcorner(x_0) \ldots (x_{(n-1)m})\left(\chi_{n-1} \vee (\beta_1) \ldots (\beta_m)\varphi\frac{x_{n_1}, \ldots, x_{nk}}{\alpha_1, \ldots, \alpha_k}\right)$$
$$\supset (x_0) \ldots (x_{(n-1)m})(\chi_{n-1} \vee (1))\urcorner \tag{8}$$

is a theorem.

(6) and (8) yield by MT111:

$$\ulcorner\mathrm{U}\chi_n \supset (x_0) \ldots (x_{(n-1)m})(\chi_{n-1} \vee (1))\urcorner,$$

and, hence, by repeated applications of MT231 and R6:

$$\ulcorner\mathrm{U}\chi_n \supset ((x_0) \ldots (x_{(n-1)m})\chi_{n-1} \vee (1))\urcorner,$$

that is,

$$\ulcorner\mathrm{U}\chi_n \supset (\mathrm{U}\chi_{n-1} \vee (1))\urcorner \tag{9}.$$

But

$$\ulcorner(9) \supset ((\mathrm{U}\chi_{n-1} \supset (1)) \supset (\mathrm{U}\chi_n \supset (1)))\urcorner \tag{10}$$

is a tautology and hence a theorem; (9) and (10) accordingly yield:

$$\ulcorner(\mathrm{U}\chi_{n-1} \supset (1)) \supset (\mathrm{U}\chi_n \supset (1))\urcorner.$$

If, as assumed, $\ulcorner\mathrm{U}\chi_{n-1} \supset (1)\urcorner$ is a theorem, $\ulcorner\mathrm{U}\chi_n \supset (1)\urcorner$ is therefore a theorem. Q.E.D.

Case 2: If there is no n such that χ_n is a tautology, then (1) is not quantificationally valid.

Proof: To show that (1) is not quantificationally valid, we shall construct an instance of (1) which is false of one universe of discourse D, the class of all natural numbers.

To construct the instance in question, let us first set up imaginary truth-tables under each χ_n, that is, under

$$\chi_1, \chi_2, \ldots, \chi_n, \ldots.$$

Each one of these truth-tables will have 2^m rows, where m is the number of different components of χ_n, and each one of these 2^m rows will assign one of the two truth-values 'T' and 'F' to the m components of χ_n. Since χ_n is not a tautology for any n, the truth-table under each χ_n will include a finite number of rows which assign 'F' to χ_n, or, to phrase matters more technically, there will be for each χ_n a finite number of truth-value

assignments to the components of χ_n which falsify χ_n. Since, however, there is an infinite set of χ_n's, there will altogether be an infinite set of truth-value assignments which falsify the χ_n's, a finite subset of this infinite set falsifying each individual χ_n.

Let us next form out of this infinite set of falsifying assignments a so-called *master assignment A* which will:

(*a*) assign one of the two truth-values 'T' and 'F' to each one of the infinitely many components which occur in the infinite set of formulae:

$$\chi_1, \chi_2, \ldots, \chi_n, \ldots,$$

and (*b*) simultaneously falsify each one of

$$\chi_1, \chi_2, \ldots, \chi_n, \ldots.$$

A may be constructed as follows. Let us first enumerate all the components of

$$\chi_1, \chi_2, \ldots, \chi_n, \ldots.$$

Let us next take the first one of these components, say ω_1, and consult each one of the falsifying assignments of

$$\chi_1, \chi_2, \ldots, \chi_n, \ldots.$$

If ω_1 is assigned 'T' in infinitely many assignments, then we assign to it the truth-value 'T' in A; otherwise, the truth-value 'F'. Let us next take the second of our components, say ω_2, and consult each one of the falsifying assignments of

$$\chi_1, \chi_2, \ldots, \chi_n, \ldots,$$

in which ω_1 has the truth-value selected for it in A. If ω_2 is assigned 'T' in infinitely many of these assignments, then we assign to it the truth-value 'T' in A; otherwise, the truth-value 'F'. And so on for each ω_i.

It is easily seen that the resulting assignment A assigns one of the two truth-values 'T' and 'F' to each one of the infinitely many components of

$$\chi_1, \chi_2, \ldots, \chi_n, \ldots.$$

It is easily seen, on the other hand, that the resulting assignment A simultaneously falsifies each one of

$$\chi_1, \chi_2, \ldots, \chi_n, \ldots.$$

Let us finally construct an instance χ'_n of χ_n by:

(*a*) replacing each sentential dummy ζ of χ_n by '$0 = 0$' or '$0 \neq 0$', depending upon whether ζ is assigned 'T' or 'F' in the master assignment A;

(*b*) replacing each formula $\ulcorner\kappa(x_{i_1}, x_{i_2}, \ldots, x_{i_p})\urcorner$ in χ_n by '$i_1 = i_1 . i_2 = i_2 . \ldots . i_p = i_p$' or its denial depending upon whether $\ulcorner\kappa(x_{i_1}, x_{i_2}, \ldots, x_{i_p})\urcorner$ is assigned 'T' or 'F' in the master assignment A.

We shall refer to the summands of the resulting χ'_n as $\psi'_1, \psi'_2, \ldots, \psi'_n$.

Since χ'_n is constructed out of χ_n in keeping with the master assignment A and A falsifies χ_n, then each one of the statements:

$$\chi'_1, \chi'_2, \ldots, \chi'_n, \ldots,$$

is false, and each one of the denials:

$$\ulcorner{\sim}\chi'_1\urcorner, \ulcorner{\sim}\chi'_2\urcorner, \ldots, \ulcorner{\sim}\chi'_n\urcorner, \ldots,$$

is true. But

$$\ulcorner{\sim}\chi'_n\urcorner$$

is equivalent by MT162 to

$$\ulcorner{\sim}\chi'_{n-1} . {\sim}\psi'_n\urcorner,$$

which implies by MT127:

$$\ulcorner{\sim}\psi'_n\urcorner;$$

$\ulcorner{\sim}\psi'_n\urcorner$ is therefore true for every n.

Now by repeated applications of MT217,

$$\ulcorner{\sim}\psi'_n \supset (\mathrm{E}\beta_1)(\mathrm{E}\beta_2)\ldots(\mathrm{E}\beta_m)$$
$${\sim}\psi'_n\frac{\beta_1\qquad,\qquad\beta_2\qquad,\ \ldots,\ \beta_m}{(n-1)m+1,\ (n-1)m+2,\ \ldots,\ nm}\urcorner$$

is a theorem; its consequent:

$$\ulcorner(\mathrm{E}\beta_1)(\mathrm{E}\beta_2)\ldots(\mathrm{E}\beta_m)\ {\sim}\psi'_n\frac{\beta_1\qquad,\qquad\beta_2\qquad,\ \ldots,\ \beta_m}{(n-1)m+1,\ (n-1)m+2,\ \ldots,\ nm}\urcorner \qquad (11),$$

is therefore true. But (11) is true for every n and, hence, for every k-tuple of natural numbers $n_1, n_2, \ldots, n_k$;

$$\ulcorner(\alpha_1)(\alpha_2)\ldots(\alpha_k)(\mathrm{E}\beta_1)(\mathrm{E}\beta_2)\ldots(\mathrm{E}\beta_m)$$
$${\sim}\psi'_n\frac{\alpha_1, \alpha_2, \ldots, \alpha_k,\qquad\beta_1\qquad,\qquad\beta_2\qquad,\ \ldots,\ \beta_m}{n_1, n_2, \ldots, n_k,\ (n-1)m+1,\ (n-1)m+2,\ \ldots,\ nm}\urcorner$$

is therefore true and, hence, by repeated applications of MT207 and MT208 so is:

$$\ulcorner{\sim}(\mathrm{E}\alpha_1)(\mathrm{E}\alpha_2)\ldots(\mathrm{E}\alpha_k)(\beta_1)(\beta_2)\ldots(\beta_m)$$
$$\psi'_n\frac{\alpha_1, \alpha_2, \ldots, \alpha_k,\qquad\beta_1\qquad,\qquad\beta_2\qquad,\ \ldots,\ \beta_m}{n_1, n_2, \ldots, n_k,\ (n-1)m+1,\ (n-1)m+2,\ \ldots,\ nm}\urcorner \qquad (12).$$

But if (12) is true, then

$$\ulcorner(\mathrm{E}\alpha_1)(\mathrm{E}\alpha_2)\ \ldots\ (\mathrm{E}\alpha_k)(\beta_1)(\beta_2)\ \ldots\ (\beta_m)$$
$$\psi'_n \frac{\alpha_1, \alpha_2, \ldots, \alpha_k, \qquad \beta_1 \qquad , \qquad \beta_2 \qquad , \ldots, \beta_m}{n_1, n_2, \ldots, n_k, (n-1)m+1, (n-1)m+2, \ldots, nm}\urcorner \tag{13}$$

is false. We have accordingly found one universe of discourse D, the class of all natural numbers, and one instance of (1), namely: (13), such that (13) is false of D. If χ_n is not a tautology for any n, (1) is therefore not quantificationally valid. Q.E.D.[12]

*48. THE QUANTIFICATIONAL CALCULUS: DECIDABILITY

In 1936 A. Church proved that there cannot exist a mechanical method for deciding whether any well-formed formula of the quantificational calculus is a theorem of the calculus or not, and hence that the quantificational calculus is undecidable; Church's argument is too advanced to be reproduced here. Though the quantificational calculus as a whole is undecidable, fragments of it, however, like the monadic quantificational calculus, are decidable.

We asserted in section 20 that if a well-formed formula φ of the monadic quantificational calculus is 2^k-valid, where k is the number of distinct predicate dummies occurring in φ, then φ is valid. A proof of this assertion follows.

MT255: If a well-formed formula φ of the monadic quantificational calculus is 2^k-valid, where k is the number of distinct predicate dummies occurring in φ, then φ is valid.

Proof: Let $\kappa_1, \kappa_2, \ldots, \kappa_k$ be, in the order of their appearance in φ, all the monadic predicate dummies of φ; let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be monadic predicates; let ψ be the result of replacing at each one of its occurrences in φ the predicate dummy κ_i ($i = 1, 2, \ldots, k$) by the predicate λ_i; and let D_1 be a non-empty universe of discourse whose members are respectively designated by 'a_1', 'a_2', . . . , 'a_n', ... We shall now show that if there is an instance, say ψ_1, of ψ (and hence of φ) which is false of D_1, then there is an instance ψ_2 of φ which is false of a 2^k-membered universe of discourse D_2.[13]

[12]The proof of case 1 is a syntactical proof, that of case 2 a semantical one. A syntactical variant of MT254 also exists, which stands to MT254 as MT183 stands to MT184. A new proof of MT254 has recently been offered by the American logician Leon Henkin.

[13]Note that if ψ does not contain any sentential dummy nor any free argument variable, then ψ_1 is simply ψ; if, however, ψ contains any sentential dummy or any free argument variable, then ψ_1 may be obtained from ψ by substituting statements for the sentential dummies of ψ and one or more of the arguments 'a_i' for the free argument variables of ψ.

Let 'A_1' designate the class of all the members of D_1 of which $\lambda_1, \lambda_2, \ldots, \lambda_k$ are true;

let 'A_2' designate the class of all the members of D_1 of which λ_1 is false, and $\lambda_2, \lambda_3, \ldots, \lambda_k$ are true;

let 'A_3' designate the class of all the members of D_1 of which λ_1 is true, λ_2 is false, and $\lambda_3, \lambda_4, \ldots, \lambda_k$ are true;

let 'A_4' designate the class of all the members of D_1 of which λ_1 and λ_2 are false, and $\lambda_3, \lambda_4, \ldots, \lambda_k$ are true;

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and let 'A_{2^k}' designate the class of all the members of D_1 of which $\lambda_1, \lambda_2, \ldots, \lambda_k$ are false;

let D_2 be the class of all A_j's ($j = 1, 2, \ldots, 2^k$); let $\mu_1, \mu_2, \ldots, \mu_k$ be monadic predicates such that $\ulcorner \mu_i(A_j) \urcorner$ is true if and only if λ_i is true of all the members of A_j; and let ψ_2 be the result of replacing at each one of its occurrences in ψ_1 the predicate λ_i by the predicate μ_i and the argument 'a_q' ($q = 1, 2, \ldots, n, \ldots$) by the argument '$A_j$' such that a_q belongs to A_j.

It follows from the construction of ψ_1 and ψ_2 that if ψ_1 is false of D_1, then ψ_2 is false of D_2. We may thus conclude that if an instance of φ is false of a non-empty universe of discourse, then an instance of φ is false of a 2^k-membered universe of discourse, or, by contraposition, that if all the instances of φ are true of all 2^k-membered universes of discourse, then all the instances of φ are true of all non-empty universes of discourse. But a formula φ is said to be 2^k-*valid* if all its instances are true of all 2^k-membered universes of discourse, and to be *valid* if all its instances are true of all non-empty universes of discourse. We may thus conclude that if φ is 2^k-valid, then φ is valid.

MT256: The monadic quantificational calculus is decidable.

Proof: According to MT255, if a formula φ of the monadic quantificational calculus is 2^k-valid, where k is the number of distinct predicate dummies occurring in φ, then φ is valid; but if φ is valid, then φ is 2^k-valid; hence φ is valid if and only if φ is 2^k-valid. According to MT254, on the other hand, φ is a theorem of the monadic quantificational calculus if and only if φ is valid; hence φ is a theorem of the monadic quantificational calculus if and only if φ is 2^k-valid. But there exists a mechanical way of deciding whether φ is 2^k-valid, the test set forth in section 20;

there accordingly exists a mechanical way of deciding whether φ is a theorem of the monadic quantificational calculus.[14]

The axioms and rules of deduction of the quantificational calculus, like the axioms and rules of deduction of the sentential calculus, are independent; this assertion will be left here without proof.

*49. THE IDENTITY CALCULUS

We conclude with a few remarks on the identity calculus:

1. The identity calculus is both relatively and absolutely consistent;
2. The identity calculus is neither relatively nor absolutely complete, but it may be shown, by an extension of Gödel's reasoning, that a formula of the identity calculus is a theorem of the identity calculus if and only if it is identically valid or identically true;
3. The identity calculus is undecidable;
4. The axioms and rules of deduction of the identity calculus are independent.

[14]The proofs of MT255 and MT256 are semantical proofs; syntactical variants of them may be found in the literature.

Exercises

CHAPTER ONE

SECTION 6:

1. Put the following statements in logical form:

(*a*) If Harry does not vote unless John does, nor John does unless Peter does, then Peter votes if Harry does.

(*b*) If Harry votes whenever John does, then Harry votes except when John fails to vote.

(*c*) Harry does not vote unless bribed to do so, while John votes whether bribed or not to do so.

SECTION 7:

2. Indicate which of the 48 tautologies listed in section 7 yield meaningful numerical formulae and which ones yield valid numerical formulae, when the four letters 'p', 'q', 'r', and 's' are interpreted as place-holders for numerals and the connectives '$\sim$', '$.$', '$\vee$', '$\supset$', '$\equiv$', and '$\not\equiv$' are respectively replaced by '$1-$', '$\times$', '$+$', '$\leq$', '$=$', and '$\neq$'.

3. The conditional schemata: '$q \supset p$', '$\sim p \supset \sim q$', and '$\sim q \supset \sim p$', are respectively called *the converse, the inverse,* and *the contrapositive of* '$p \supset q$'. Indicate which of the following pairs of conditional schemata are logically equivalent:

(*a*) '$p \supset q$' and its converse;
(*b*) '$p \supset q$' and its inverse;
(*c*) '$p \supset q$' and its contrapositive;
(*d*) '$q \supset p$' and its contrapositive;
(*e*) '$q \supset p$' and its inverse;
(*f*) '$\sim p \supset \sim q$' and its inverse.

SECTION 8:

4. On page 30 substitution and interchange are shown to differ in one respect; indicate another respect in which they also differ.

SECTION 9:

5. Following the methods of section 9, deduce, whenever possible, the last statement of each group from those which precede it. In setting up deductions, use the capital letters appended as shorthands for statements.

(*a*) Harry does not vote unless John does. John does not vote unless Peter does. ∴ Peter votes if Harry does. (H, J, P)

(*b*) Harry votes whenever John does. ∴ Harry votes except when John fails to vote. (H, J)

(*c*) Paul is neurotic or is irresponsible. Paul is neurotic. ∴ Paul is not irresponsible. (N, I)

(*d*) If Peter gets a car for the summer, then he goes West. If John gets a car for the summer, then he goes East. Peter or John gets a car for the summer. Peter does not go West. ∴ John goes East. (P, W, J, E)

(*e*) The moon is smaller than the sun. ∴ If the moon is made up of green cheese, then it is smaller than the sun. (S, M)

(*f*) One may enjoy *Ulysses* if one knows Homer or Dublin. ∴ One may enjoy *Ulysses* if one knows Dublin. (E, H, D)

(*g*) If A's or B's team wins, then C's or D's team is eliminated from the league. A's team wins, but D's is not eliminated from the league. ∴ If B's team wins, then C's team is eliminated from the league. (A, B, C, D)

(*h*) If Paul resigns, then Peter or John gets promoted. If John gets promoted, then he buys a new home if he is offered one at a decent price. Peter does not get promoted. ∴ Paul does not resign or John buys a new home. (R, P, J, B, O)

(*i*) Mary does not cut biology or she cuts chemistry. Mary cuts chemistry, but does not cut physics. ∴ Mary does not cut biology or physics. (B, C, P)

SECTION 10:

6. Compute the truth-values of the following schemata:

(*a*) $p \supset (p \vee q)$;

(*b*) $(p \supset (q \supset r)) \equiv ((p \,.\, q) \supset r)$;

(*c*) $(p \supset (q \,.\, r)) \equiv ((p \supset q) \vee (p \supset r))$.

7. Test the groups of statements (*a*)–(*f*) of exercise 5 as to sentential deductibility. Note: When setting up a truth-table for a statement, one may list to the left of the statement all its atomic components and proceed from that point on as with a schema.

8. Test the following groups of statements as to sentential consistency:

(*a*) If salaries do not rise, then incomes do not rise. If salaries rise, then profits do not rise. Incomes rise. Profits rise. (S, I, P)

(*b*) A wins if B loses. A does not win. B loses. (W, L)

(*c*) If A and B graduate, then C flunks out of college. If D flunks out of college, then B graduates. D flunks out of college, but A graduates. C does not flunk out of college. (A, B, C, D)

9. Deduce from the statements listed in 8(*c*) each one of the following pairs of statements: 1. 'A' and '∼A'; 2. 'B' and '∼B'; 3. 'C' and '∼C'; 4. 'D' and '∼D'.

10. Test the following schemata as to sentential validity:

(*a*) $\sim p \equiv (p \downarrow p)$;

(*b*) $(p \,.\, q) \equiv (\sim p \downarrow \sim q)$;

(*c*) $(p \vee q) \equiv \sim(p \downarrow q)$;

(*d*) $\sim p \equiv (p \mid p)$;

(*e*) $(p \,.\, q) \equiv \sim(p \mid q)$;

(*f*) $(p \vee q) \equiv (\sim p \mid \sim q)$.

SECTION 11:

11. Set up defining tables for '$\sim_4$', '$._4$', '$\vee_4$', '$\supset_4$', '$\equiv_4$', and '$\not\equiv_4$'.

12. Compute the truth-values of the following three-valued schemata:

(*a*) $p \supset_3 p$;

(*b*) $(p \,._3\, q) \equiv_3 (q \,._3\, p)$;

(*c*) $(p \supset_3 (q \supset_3 r)) \equiv_3 ((p \,._3\, q) \supset_3 r)$;

(*d*) $p_3 \supset_3 (\sim_3 p \supset_3 p)$;
(*e*) $(\sim_3 p \supset_3 p) \supset_3 p$.

SECTION 12:

13. It is shown on page 47 that the modal compounds (1)–(4) may be expressed in terms of '$\sim$' and '$\Diamond$'. Show that the same four compounds may also be expressed in terms of '$\sim$' and:

(*a*) '$\Box$' ('It is necessary that');
(*b*) '$\triangle$' ('It is not necessary that');
(*c*) '∇' ('It is not possible that').

14. Using the definition of an M-schema on page 50, show that M1–M20 are M-schemata.

15. Find the M-schemata corresponding to the following metalogical statements:

(*a*) If a given statement logically implies another statement, then the denial of the latter logically implies the denial of the former.

(*b*) Two statements are logically consistent if and only if neither one logically implies the denial of the other.

(*c*) If a given statement is not logically consistent with itself, then it is not logically consistent with any other statement.

(*d*) A statement which is logically false logically implies any other statement.

(*e*) A statement which is logically true is logically implied by any other statement.

CHAPTER TWO

SECTION 13:

1. Put statements (*a*)–(*i*) from exercise 5, chapter one, in logical form.

SECTION 14:

2. Put statements (*a*)–(*m*) from exercise 8 below in logical form.
3. Form all the closures of '$F(w)$', '$F(w,x)$', '$F(w,x,y)$', and '$F(w,x,y,z)$'.

SECTION 17:

4. Compute the truth-values of Q30a–Q36i.
5. Identify the mood and figure of the following eight syllogisms:

(*a*) All politicians are liars. Some Communists are politicians. ∴ Some Communists are liars. (P, L, C)

(*b*) All tenors are pretentious. No bass is a tenor. ∴ No bass is pretentious. (T, P, B)

(*c*) All C. I. O. members are Democrats. All Southerners are Democrats. ∴ All Southerners are C. I. O. members. (M, D, S)

(*d*) No murder story is dull. All textbooks are dull. ∴ No textbook is a murder story. (M, D, T)

(*e*) No bachelor is married. Some married people are happy. ∴ Some happy people are not bachelors. (B, M, H)

(*f*) Some psychologists do not favor divorce. All psychologists are scientists. ∴ Some scientists do not favor divorce. (P, F, S)

(*g*) Some poets do not like Pound. All Pound lovers are eccentrics. ∴ Some eccentrics are not poets. (P, L, E)

(*h*) No Spaniard is cold-blooded. All Spaniards are bull-fighters. ∴ Some bull-fighters are not cold-blooded. (S, C, B)

6. Produce false instances of the converses of Q36d–Q36h.

7. Put statements (*a*)–(*e*) from exercise 11 below in logical form.

SECTION 18:

8. Determine which ones of the statements φ' constructed in examples 1, 2, and 3, on pages 86–87, may respectively count as instances of Q35a, Q33, and Q39a.

SECTION 19:

9. Following the methods of section 19, deduce, whenever possible, the last statement of each group from those which precede it. In setting up deductions, use the capital letters appended as shorthands for predicates and the small letters appended as shorthands for arguments.

(*a*)–(*h*): cf. syllogisms listed in problem 5.

(*i*) Everything hermetic appeals to crackpots. Nothing which appeals to crackpots is art. Some contemporary poetry is meaningless. ∴ Some contemporary poetry is not art. (M, C, A, P)

(*j*) All the fourth-graders who started the row are spoiled brats who deserve a spanking. ∴ If no spoiled brat deserves a spanking, no fourth-grader started the row. (F, S, C, D)

(*k*) Some paintings in this room are Monets or Manets. ∴ Some paintings in this room are Monets. (P, O, A)

(*l*) All Americans who appreciate good music listen to the N. B. C. Symphony. John does not listen to the N. B. C. Symphony. ∴ John does not appreciate good music. (A, M, L, j)

(*m*) All the members of the club are freshmen or sophomores. Some members of the club are not freshmen. ∴ Some members of the club are sophomores. (M, F, S)

(*n*) Anyone who has heard Berg and Webern likes Berg. Sally has heard Berg, but does not like it. ∴ Someone has not heard Webern. (B, W, L, s)

10. The following nine premises, due to Lewis Carroll: 'The only animals in this house are cats', 'Every animal is suitable for a pet, that loves to gaze at the moon', 'When I detest an animal, I avoid it', 'No animals are carnivorous, unless they prowl at night', 'No cat fails to kill mice', 'No animals ever take to me, except what are in this house', 'Kangaroos are not suitable for pets', 'None but carnivora kill mice', 'I detest animals that do not take to me', are known to imply the conclusion: 'I always avoid a kangaroo', when supplemented with a tenth premise. Find such a premise.

11. Given:

1. $(x)(y)F(x,y)$,	5. $(y)(x)F(x,y)$,
2. $(x)(\mathrm{E}y)F(x,y)$,	6. $(y)(\mathrm{E}x)F(x,y)$,
3. $(\mathrm{E}x)(y)F(x,y)$,	7. $(\mathrm{E}y)(x)F(x,y)$,
4. $(\mathrm{E}x)(\mathrm{E}y)F(x,y)$,	8. $(\mathrm{E}y)(\mathrm{E}x)F(x,y)$,
9. $(x)(y)F(y,x)$,	13. $(y)(x)F(y,x)$,
10. $(x)(\mathrm{E}y)F(y,x)$,	14. $(y)(\mathrm{E}x)F(y,x)$,
11. $(\mathrm{E}x)(y)F(y,x)$,	15. $(\mathrm{E}y)(x)F(y,x)$,
12. $(\mathrm{E}x)(\mathrm{E}y)F(y,x)$,	16. $(\mathrm{E}y)(\mathrm{E}x)F(y,x)$,

list all pairs of schemata such that:

(*a*) the first member of the pair logically implies the second;

(*b*) the first member of the pair is logically equivalent to the second.

12. Following the methods of section 19, deduce, whenever possible, the last statement of each group from those which precede it:

(*a*) Those who can get along with everyone in their family can get along with everyone in their community. ∴ If there is someone in his community with whom John cannot get along, then there is someone in his family with whom he cannot get along. (G, F, C, j)

(*b*) No man on the staff can outdo everyone in the factory. John is in the factory. Peter cannot outdo John. ∴ Peter is on the staff. (S, O, F, j, p)

(*c*) If a person is younger than another person who is, in turn, younger than a third person, then the first is younger than the third. Nobody is younger than himself. ∴ If a person is younger than a second person, then the second person is not younger than the first. (Y)

(*d*) No member of A's team can beat every member of C's team. ∴ Some member of C's team can beat some member of A's team. (A, B, C)

(*e*) From the two premises:

1. A thing is part of another thing if and only if whatever is part of the first is also part of the second,
2. A thing overlaps another thing if and only if the two things have a part in common,

deduce the following conclusions:

3. Everything is part of itself and overlaps itself;
4. If a thing overlaps another thing, then the second overlaps the first;
5. If a thing is part of another thing which, in turn, is part of a third thing, then the first is part of the third;
6. A thing overlaps whatever it is part of;
7. A thing is part of another thing if and only if whatever overlaps the second also overlaps the first;
8. If a thing overlaps all the parts of another thing, then the second is part of the first;
9. If a thing is part of another thing and overlaps a third, then the second also overlaps the third.

Section 20:

13. (*a*) Produce an instance φ of '$(\mathrm{E}x)F(x) \supset (x)F(x)$' such that φ is true of all universes of discourse; (*b*) Produce an instance φ of '$(\mathrm{E}x)F(x) \,.\, (\mathrm{E}x){\sim}F(x)$' such that φ is false of all universes of discourse.

14. Find an n such that '$(\mathrm{E}x)F(x) \supset (x)F(x)$' is not n-valid and an n such that '$(\mathrm{E}x)F(x) \,.\, (\mathrm{E}x){\sim}F(x)$' is not n-valid.

15. Prove that Q30a–Q39c are 2-valid.

16. Prove that Q30a–Q371 are valid.

17. Prove that '$(x){\sim}F(x,x) \,.\, ((x)(y)(z)((F(x,y) \,.\, F(y,z)) \supset F(x,z)) \,.\, (x)(\mathrm{E}y)F(x,y))$' is not 2-valid.

CHAPTER THREE

Section 23:

1. Reduce the primitive signs of the sentential and of the quantificational calculus to two finite sets of signs.

2. Show by means of examples why of the two criteria of definability given on page 106 (b) is weaker than (a).

3. Define '$\vee$' in terms of '$\supset$'; '$\vee$' in terms of '$\subset$'; '$\sim$' and '$\vee$' in terms of '$|$'; and '$\sim$' and '$\vee$' in terms of '$\downarrow$'.

4. Define in terms of '$\sim$' and '$\vee$' all the binary modes of sentential composition listed in section 10, chapter one.

Section 24:

5. Prove that schemata T1–T9c of section 7, chapter one, and schemata Q30a–Q35b of section 17, chapter two, are well-formed.

Section 25:

6. Prove metatheorem A on page 114.

Section 28:

7. Prove metatheorems MT109–MT114.
8. Prove metatheorems MT136–MT170.
9. Set up a complete list of the metatheorems needed to prove:

(*a*) the rule of Adjunction;
(*b*) the rule of Conditionalization;
(*c*) the rule of Interchange.

10. Following the directions given in the proof of R5,

(*a*) turn the derivation of $\ulcorner(\varphi \supset \psi) \supset (\psi \vee \chi)\urcorner$ from the assumption formula φ, on pages 131–132, into a proof of $\ulcorner\varphi \supset ((\varphi \supset \psi) \supset (\psi \vee \chi))\urcorner$;

(*b*) turn the derivation of χ from the three assumption formulae $\ulcorner(\varphi \,.\, \psi) \supset \chi\urcorner$, φ, and ψ, on page 129, into a proof of $\ulcorner((\varphi \,.\, \psi) \supset \chi) \supset (\varphi \supset (\psi \supset \chi))\urcorner$.

11. Following the directions given in the proof of R6, show that:

(*a*) $(p \equiv (q \,.\, r)) \supset ((s \vee p) \equiv (s \vee (q \,.\, r)))$

and

(*b*) $((q \,.\, r) \equiv p) \supset ((s \vee (q \,.\, r)) \equiv (s \vee p))$

are theorems.

Section 29:

12. Prove metatheorems MT207–MT209, MT212–MT214, MT216, MT218, MT220–MT236, MT238–MT241, MT243, and MT244.

13. Prove the rule of Argument Substitution (R3b) and the rule of Predicate Substitution (R3c) stated on pages 142–143.

14. (*a*) The first eight steps of the proof of MT204 constitute a derivation of $\ulcorner(\alpha)\psi\urcorner$ from the two assumption formulae $\ulcorner(\alpha)(\varphi \supset \psi)\urcorner$ and $\ulcorner(\alpha)\varphi\urcorner$; following the directions given in the proof of R5, turn these eight steps into a proof of $\ulcorner(\alpha)(\varphi \supset \psi) \supset ((\alpha)\varphi \supset (\alpha)\psi)\urcorner$.

(*b*) The first eight steps of the proof of MT205 constitute a derivation of $\ulcorner(\alpha)\varphi \,.\, (\alpha)\psi\urcorner$ from the assumption formula $\ulcorner(\alpha)(\varphi \,.\, \psi)\urcorner$; following the directions given in the proof of R5, turn these eight steps into a proof of $\ulcorner(\alpha)(\varphi \,.\, \psi) \supset ((\alpha)\varphi \,.\, (\alpha)\psi)\urcorner$.

15. Following the directions given in the proof of R6, show that:

(*a*) $(x)((w)(z)F(w,z) \equiv G(x)) \supset ((x)(H(x) \supset (w)(z)F(w,z)) \equiv (x)(H(x) \supset G(x)))$

and

(*b*) $(x)((y)(F(y) \supset p) \equiv G(x)) \supset ((x)(H(x) \supset (y)(F(y) \supset p)) \equiv (x)(H(x) \supset G(x)))$

are theorems.

16. Following the directions given in the proof of R7, show that:

(*a*) if '$(x)(y)(z)F(x,y,z)$' is a theorem, then so is '$(y)(z)(x)F(y,z,x)$';
(*b*) if '$(x)(F(x) \supset (x)(y)G(x,y))$' is a theorem, then so is '$(y)(F(y) \supset (y)(x)G(y,x))$'.

17. Set up a list of all the metatheorems needed to prove:

(*a*) the rule of Conditionalization;
(*b*) the rule of Interchange;
(*c*) the rule of Relettering.

Section 30:

18. Show that version II of the sentential calculus is equivalent to Rosser's. Plan: (*a*) Deduce MB100–MB102 and the biconditional $\ulcorner(\varphi \vee \psi) \equiv \sim(\sim\varphi \,.\, \sim\psi)\urcorner$ in version II of the sentential calculus;

(*b*) Deduce MA100–MA103, the rule of Interchange, and the biconditional $\ulcorner(\varphi \,.\, \psi) \equiv \sim(\sim\varphi \vee \sim\psi)\urcorner$ in Rosser's version of the sentential calculus.

19. Show that version II of the sentential calculus is equivalent to Łukasiewicz's.

20. Show that version II of the sentential calculus is equivalent to Wajsberg's.

21. Show that our version of the quantificational calculus is equivalent to Hilbert's. Note: 18 and 21 are routine problems; 19 and, especially, 20 are intended for the advanced reader.

Section 31:

22. (*a*) Formulate and prove in version I of the sentential calculus rules of deduction corresponding to CnE, AI, AE, NI, NE, BI, and BE. (*b*) Formulate and prove in our version of the quantificational calculus rules of deduction corresponding to AE, NI, BI, UE, EI, and EE.

23. Prove by Gentzen's method tautologies T1–T24c of section 7. Plan (*a*) Treat $\ulcorner(\varphi \equiv \psi)\urcorner$ as a paraphrase of $\ulcorner((\varphi \supset \psi) \,.\, (\psi \supset \varphi))\urcorner$. Plan (*b*) Treat '$\equiv$' as a primitive sign and use derivation schemata BI and BE.

24. Prove by Gentzen's method schemata Q30a–Q39c of section 17; follow plans (*a*) and (*b*) of problem 23.

25. Using NE′ in place of NE, prove by Gentzen's method all the I-valid schemata listed in section 21.

CHAPTER FOUR

Section 32:

1. Deduce MA301 from MA301′.
2. Prove T305, MT307, and MT308.

Section 33:

3. Find a recursive definition of $\ulcorner(\underset{n}{\mathrm{EM}}\alpha)\varphi\urcorner$ independent of D10.

4. Find a recursive definition of $\ulcorner(\underset{n}{\mathrm{EE}}\alpha)\varphi\urcorner$ independent of D10 and D11.

5. Prove MT322–MT324:

(*a*) for the cases where $n = 1, 2, 3$;
(*b*) for any n.

Section 34:

6. Produce factual instances of the two abstracts '$\hat{x}(F(x) \equiv G(x))$' and '$\hat{x}(F(x) \supset G(x))$', and prove the two conditionals '$(y)((\hat{x}F(x) = \hat{x}G(x)) \supset y \in \hat{x}(F(x) \equiv G(x)))$' and '$(y)((\hat{x}F(x) \subset \hat{x}G(x)) \supset y \in \hat{x}(F(x) \supset G(x)))$'.

7. Using 2 circles A′ and B′ as geometrical representations of two classes A and B, portray the following:

(*a*) $\mathrm{A} = \mathrm{B}$;
(*b*) $\mathrm{A} \subset \mathrm{B} \,.\, \mathrm{A} \neq \mathrm{B}$;
(*c*) $\mathrm{A} \cap \mathrm{B} = \Lambda$;
(*d*) $\mathrm{A} \cap \mathrm{B} \neq \Lambda$;
(*e*) $\mathrm{A} \cap \bar{\mathrm{B}} = \Lambda$;
(*f*) $\mathrm{A} \cap \bar{\mathrm{B}} = \Lambda \,.\, \mathrm{A} \neq \mathrm{B}$;
(*g*) $\mathrm{A} \cap \bar{\mathrm{B}} = \Lambda \,.\, \mathrm{B} \cap \bar{\mathrm{A}} = \Lambda$.

8. Using a rectangle C′ as a geometrical representation of the universal class V and two areas A′ and B′ of C′ as geometrical representations of two classes A and B, portray the following:

(*a*) $\mathrm{A} = \bar{\mathrm{B}}$;
(*b*) $\bar{\mathrm{A}} \cup \mathrm{B} = \mathrm{V}$;
(*c*) $\bar{\mathrm{A}} \cup \mathrm{B} = \mathrm{V} \,.\, \mathrm{A} \neq \mathrm{B}$;
(*d*) $\bar{\mathrm{A}} \cup \mathrm{B} = \mathrm{V} \,.\, \bar{\mathrm{B}} \cup \mathrm{A} = \mathrm{V}$;
(*e*) $\mathrm{A} \cap \mathrm{B} = \mathrm{V}$.

9. Prove MT334, MT336–MT344, and T345.

SECTION 35:

10. Prove T350–T360.
11. Prove assertions (*a*) and (*b*) on page 174.
12. Form the monadic associates of T380–T391.
13. Test the monadic associates of T380-T391 as to quantificational validity.

SECTION 36:

14. Prove T406–T414.

SECTION 37:

15. Produce factual instances of the two abstracts '$\hat{x}\hat{y}(F(x,y) \equiv G(x,y))$' and '$\hat{x}\hat{y}(F(x,y) \supset G(x,y))$'.
16. Prove MT430–T446.

SECTION 38:

17. Prove T465–T475.
18. Prove that the class associate of a Boolean relation formula φ is a theorem if and only if φ is a theorem.
19. Prove MT495.

SECTION 39:

20. Prove MT500–T502, T504–T506, MT510–T517, and MT520–T528.
21. Prove the following four statements: '$(x)(\dot{\mathrm{V}}``\{x\} = \mathrm{V})$', '$(x)(\dot{\Lambda}``\{x\} = \Lambda)$', '$(y)(\check{\mathrm{V}}``\{y\} = \mathrm{V})$, and '$(y)(\check{\Lambda}``\{y\} = \Lambda)$'.
22. Prove the following four statements: $\dot{\mathrm{V}}``\mathrm{V} = \mathrm{V}$', '$\check{\mathrm{V}}``\mathrm{V} = \mathrm{V}$', '$\dot{\Lambda}``\mathrm{V} = \Lambda$', and '$\check{\mathrm{V}}``\mathrm{V} = \Lambda$'.

SECTION 40:

23. Prove T550–T551 and T553–T556.
24. Let 'R', I', and 'NR' respectively stand for 'Reflexive', 'Irreflexive', and 'Non-reflexive'; let 'S', 'A', and 'NS' respectively stand for 'Symmetrical', 'Asym-

metrical', and 'Non-symmetrical'; and let' T', 'IT', and 'NT' respectively stand for 'Transitive', 'Intransitive', and 'Non-transitive'.

(*a*) Strike out from the following list every triple of incompatible properties:

R, S, T,	R, A, T,	R, NS, T,
R, S, IT,	R, A, IT,	R, NS, IT,
R, S, NT,	R, A, NT,	R, NS, NT.

(*b*) Perform the same operation on the nine triples of properties which result from substituting 'I' for 'R' in (*a*).

(*c*) Perform the same operation on the nine triples of properties which result from substituting 'NR' for 'R' in (*a*).

25. Enumerate all the binary modes of sentential composition which are: (*a*) reflexive, (*b*) irreflexive, (*c*) non-reflexive.

26. Enumerate all the binary modes of sentential composition which are: (*a*) symmetrical, (*b*) asymmetrical, (*c*) non-symmetrical.

27. Enumerate all the binary modes of sentential composition which are: (*a*) transitive, (*b*) intransitive, (*c*) non-transitive.

Section 41:

28. Prove T561–T572.

CHAPTER FIVE

Section 43:

1. Following the instructions given in the proof of MT181, show that:

(*a*) $(p \,.\, q) \supset (p \supset (q \supset p))$,
(*b*) $(\sim p \,.\, q) \supset (p \supset (q \supset p))$,
(*c*) $(p \,.\, \sim q) \supset (p \supset (q \supset p))$,
(*d*) $(\sim p \,.\, \sim q) \supset (p \supset (q \supset p))$,

are theorems.

2. Following the instructions given in the proof of MT181, show that:

(*a*) $(p \,.\, q \,.\, r) \supset (((p \supset q) \,.\, (q \supset r)) \supset (p \supset r))$,
(*b*) $(\sim p \,.\, q \,.\, r) \supset (((p \supset q) \,.\, (q \supset r)) \supset (p \supset r))$,
(*c*) $(p \,.\, \sim q \,.\, r) \supset (((p \supset q) \,.\, (q \supset r)) \supset (p \supset r))$,
(*d*) $(\sim p \,.\, \sim q \,.\, r) \supset (((p \supset q) \,.\, (q \supset r)) \supset (p \supset r))$,
(*e*) $(p \,.\, q \,.\, \sim r) \supset (((p \supset q) \,.\, (q \supset r)) \supset (p \supset r))$,
(*f*) $(\sim p \,.\, q \,.\, \sim r) \supset (((p \supset q) \,.\, (q \supset r)) \supset (p \supset r))$,
(*g*) $(p \,.\, \sim q \,.\, \sim r) \supset (((p \supset q) \,.\, (q \supset r)) \supset (p \supset r))$,
(*h*) $(\sim p \,.\, \sim q \,.\, \sim r) \supset (((p \supset q) \,.\, (q \supset r)) \supset (p \supset r))$,

are theorems.

3. Following the instructions given in the proof of MT182, show that:

(*a*) $p \vee \sim p$,
(*b*) $((p \supset q) \,.\, p) \supset q$,
(*c*) $((p \supset q) \,.\, (q \supset r)) \supset (p \supset r)$,

are theorems.

SECTION 44:

4. Following the instructions given in the proof of MT187, show that version I of the sentential calculus becomes relatively inconsistent if '$((p \,.\, q) \supset r) \supset (p \supset r)$' is appointed as an extra axiom.

5. Following the instructions given in the proof of MT188, show that version I of the sentential calculus becomes absolutely inconsistent if '$((p \,.\, q) \supset r) \supset (p \supset r)$' is appointed as an extra axiom.

6. (*a*). Prove the converse of R5 on page 130, namely: If $\ulcorner \varphi_1 \supset (\varphi_2 \supset (\ldots \supset (\varphi_n \supset \psi) \ldots)) \urcorner$ is a theorem, then $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$.

(*b*). Using (*a*) prove the following metatheorem: $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ if and only if $\varphi_1, \varphi_2, \ldots, \varphi_n$ logically imply ψ.

(*c*). Using (*b*) prove the two metatheorems given under (*a*) and (*b*) on pages 22–23.

7. Fill in the missing details of the proof of MT190.

SECTION 47:

8. Prove lemma I on page 210.

Hint: Three cases have to be considered: (*a*) the case where a formula φ contains two or more occurrences of the same quantifier; (*b*) the case where φ has a component of the form $\ulcorner(Q\alpha)\psi\urcorner$, where α is not free in ψ; (*c*) the case where φ has a component of the form $\ulcorner\sim(Q\alpha)\psi\urcorner$, $\ulcorner(Q\alpha)\psi \vee \chi\urcorner$, or $\ulcorner\psi \vee (Q\alpha)\chi\urcorner$.

9. Prove that:

(*a*) '$(x)F(x) \supset (\mathrm{E}x)F(x)$' is a theorem if and only if '$(\mathrm{E}x)(\mathrm{E}y)(F(y) \supset F(x))$, is a theorem;

(*b*) '$(x)(F(x) \supset G(x)) \supset ((x)F(x) \supset (x)G(x))$' is a theorem if and only if '$(\mathrm{E}x)(\mathrm{E}y)(z)((F(x) \supset G(x)) \supset (F(z) \supset G(y)))$' is a theorem.

10. Following the instructions given in the proof of MT253 show that schemata Q32a, Q35b, Q36a–Q36c, Q37i–Q37l, and Q39a–Q39c are theorems.

Note: By strategic relettering and confinement of quantifiers the reader should hit upon Skolem normal forms of the above twelve schemata without perusing a proof of lemma II.

11. List all implicit uses of R1 in the proof of MT253.

12. (*a*). Prove the following converse of R5 on page 143: If $\ulcorner \varphi_1 \supset (\varphi_2 \supset (\ldots \supset (\varphi_n \supset \psi) \ldots)) \urcorner$ is a theorem, then $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$.

(*b*). Using (*a*) prove the following metatheorem: If $\varphi_1, \varphi_2, \ldots, \varphi_n$ have no free argument variables, then $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ if and only if $\varphi_1, \varphi_2, \ldots, \varphi_n$ logically imply ψ.

(*c*). Using (*b*) prove the two metatheorems given under (*a*) and (*b*) on page 84.

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